

Periodic integral operators over Cayley-Dickson algebras and spectra

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Abstract

Periodic integral operators over Cayley-Dickson algebras related with integration of PDE are studied. Their continuity and spectra are investigated.¹

1 Introduction

Integral operators over Cayley-Dickson algebras are useful for integration of linear and non-linear partial differential equations [26, 27].

Sections 2 and 3 are devoted to spectra of periodic integral operators over Cayley-Dickson algebras, that can be used for analysis of solutions in a bounded domain or of periodic solutions on the Cayley-Dickson algebra of PDE including that of non-linear. This is actual, because spectra of operators are used for solutions of partial differential equations, for example, with the help of the inverse scattering problem method (see also [1]). Moreover, hypercomplex analysis is fast developing and also in relation with problems of theoretical and mathematical physics and of partial differential equations [4, 9, 11]. Cayley-Dickson algebras are used not only in mathematics, but also in applications [6, 12, 10, 15, 16].

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Analysis over Cayley-Dickson algebras was developed as well [18, 19, 20, 21, 22]. This paper continuous previous articles and uses their results [23, 24, 25, 26, 27]. Notations and definitions of papers [18, 19, 20, 21, 22] are used below. The main results of this paper are obtained for the first time.

2 Periodic integral operators over Cayley-Dickson algebras

1. Notation. Let X be a Banach space over the Cayley-Dickson algebra \mathcal{A}_v with $2 \leq v$, $v \in \mathbf{N}$. Let also \mathcal{A}_w be the Cayley-Dickson subalgebra $\mathcal{A}_w \subseteq \mathcal{A}_v$, where $2 \leq w \leq v$.

The Cayley-Dickson algebra \mathcal{A}_v has the real shadow \mathbf{R}^{2^v} . On this real shadow take the Lebesgue measure μ so that $\mu(\prod_{j=1}^{2^w} [a_j, a_j + 1]) = 1$ for each $a_j \in \mathbf{R}$. This measure induces the Lebesgue measure on \mathcal{A}_w denoted also by μ . A subset A in \mathcal{A}_w is called μ -null if a Borel subset B in \mathcal{A}_w exists so that $A \subseteq B$ and $\mu(B) = 0$. The Lebesgue measure is defined on the completion $\mathcal{B}_\mu(\mathcal{A}_w)$ of the Borel σ -algebra $\mathcal{B}(\mathcal{A}_w)$ on \mathcal{A}_w by μ -null sets.

If $1 \leq p < \infty$, $L^p(\mathcal{A}_w, X)$ will denote the norm completion of a space of all μ -measurable step functions $f : \mathcal{A}_w \rightarrow X$ for which the norm

$$(1) \quad \|f\|_p := \sqrt[p]{\int_{\mathcal{A}_w} \|f(z)\|_X^p \mu(dz)}$$

is finite, where

$$f = \sum_{k=1}^n b_k \chi_{B_k}$$

is a step function so that $b_k \in X$ and $B_k \in \mathcal{B}_\mu(\mathcal{A}_w)$ for each $k = 1, \dots, n$; $B_k \cap B_j = \emptyset$ for each $j \neq k$; $\|x\|_X$ denotes the norm of a vector x in X , certainly, the norm is non-negative, $0 \leq \|x\|_X$, n is a natural number. If $p = \infty$, the norm is given by the formula:

$$(2) \quad \|f\|_\infty := \text{ess}_\mu \sup_{z \in \mathcal{A}_w} \|f(z)\|_X.$$

Then a Banach space $L_q(L^p(\mathcal{A}_w, X))$ of all bounded \mathbf{R} -homogeneous \mathcal{A}_v additive operators $T : L^p(\mathcal{A}_w, X) \rightarrow L^p(\mathcal{A}_w, X)$ is considered. Let $K : \mathcal{A}_w^2 \rightarrow L_q(X)$ be a strongly measurable operator valued mapping, that is a mapping

$g(t, s) := K(t, s)y : \mathcal{A}_w^2 \rightarrow X$ is $(\mathcal{B}_{\mu^2}(\mathcal{A}_w^2), \mathcal{B}(X))$ measurable for each vector $y \in X$, i.e. $g^{-1}(Q) \in \mathcal{B}_{\mu^2}(\mathcal{A}_w^2)$ for each Borel subset $Q \in \mathcal{B}(X)$, where μ^2 is the Lebesgue measure on \mathcal{A}_w^2 .

In the paper [27] the following theorem about first order partial differential operators with variable \mathcal{A}_v coefficients was demonstrated.

Theorem. *Suppose that a first order partial differential operator Υ is given by the formula*

$$(i) \quad \Upsilon f = \sum_{j=0}^n (\partial f / \partial z_j) \phi_j^*(z),$$

where $\phi_j(z) \neq \{0\}$ for each $z \in U$ and $\phi_j(z) \in C^0(U, \mathcal{A}_v)$ for each $j = 0, \dots, n$ such that $\text{Re}(\phi_j(z)\phi_k^*(z)) = 0$ for each $z \in U$ and each $0 \leq j \neq k \leq n$, where a domain U satisfies Conditions 2.1.1(D1, D2), ${}_0z$ is a marked point in U , $1 < n < 2^v$, $2 \leq v$. Suppose also that a system $\{\phi_0(z), \dots, \phi_n(z)\}$ is for $n = 2^v - 1$, or can be completed by Cayley-Dickson numbers $\phi_{n+1}(z), \dots, \phi_{2^v-1}(z)$, such that (α) $\text{alg}_{\mathbf{R}}\{\phi_j(z), \phi_k(z), \phi_l(z)\}$ is alternative for all $0 \leq j, k, l \leq 2^v - 1$ and (β) $\text{alg}_{\mathbf{R}}\{\phi_0(z), \dots, \phi_{2^v-1}(z)\} = \mathcal{A}_v$ for each $z \in U$. Then a line integral $\mathcal{I}_{\Upsilon} : C^0(U, \mathcal{A}_v) \rightarrow C^1(U, \mathcal{A}_v)$, $\mathcal{I}_{\Upsilon} f(z) := \Upsilon \int_{{}_0z}^z f(y) dy$ on $C^0(U, \mathcal{A}_v)$ exists so that

$$(ii) \quad \Upsilon \mathcal{I}_{\Upsilon} f(z) = f(z)$$

for each $z \in U$; this anti-derivative is \mathbf{R} -linear (or \mathbf{H} -left-linear when $v = 2$). If there is a second anti-derivative $\mathcal{I}_{\Upsilon,2} f(z)$, then $\mathcal{I}_{\Upsilon} f(z) - \mathcal{I}_{\Upsilon,2} f(z)$ belongs to the kernel $\ker(\Upsilon)$ of the operator Υ .

For a first order partial differential operator $\sigma = \Upsilon$ over \mathcal{A}_w with constant or variable coefficients consider the antiderivative operator $\sigma \int$ on \mathcal{A}_w . Put

$$(3) \quad (Bx)(t) := \sigma \int_{-\infty}^{\infty} K(t, s)x(s)ds$$

whenever this integral converges in the weak sense as

$$(4) \quad \sigma \int_{-\infty}^{\infty} u[K(t, s)x(s)yds] := \lim_{a \rightarrow a_{\alpha}, b \rightarrow b_{\alpha}} \sigma \int_{\gamma^{\alpha}|_{[a, b]}} u[K(t, s)x(s)yds] \in \mathcal{A}_v$$

for each $y \in X$ and right \mathcal{A}_v linear continuous functional $u \in L_r(X, \mathcal{A}_v) = X_r^*$, where $x \in L_q(L^p(\mathcal{A}_w, X), L^{p'}(\mathcal{A}_w, X))$ so that $(xf)(s) = x(s)f(s)$ for each $f \in L^p(\mathcal{A}_w, X)$, $x(s) \in L_q(X)$ for every $s \in \mathcal{A}_w$,

$$\lim_{a \rightarrow a_{\alpha}} \gamma^{\alpha}(t) = \infty \text{ and } \lim_{b \rightarrow b_{\alpha}} \gamma^{\alpha}(t) = \infty,$$

$a_\alpha < b_\alpha$, $\hat{\mathcal{A}}_w$ is the one-point (Alexandroff) compactification of the Cayley-Dickson algebra as the topological space, $\infty = \hat{\mathcal{A}}_w \setminus \mathcal{A}_w$, $\alpha \in \Lambda$. The integral in Formula (4) reduces to the integral described in §4.2.5 [22]. Consider a periodic integral kernel

$$(5) \quad K(t, s) = K(t + p_j \omega_j i_j, s + p_j \omega_j i_j),$$

also $\phi_j(s + p_j \omega_j i_j) = \phi_j(s)$ for μ almost every Cayley-Dickson numbers $t, s \in \mathcal{A}_w$ for all integers p_j , where $\omega_j > 0$ is a period by z_j , $z = z_0 i_0 + \dots + z_{2^w-1} i_{2^w-1} \in \mathcal{A}_w$, $z_0, \dots, z_{2^w-1} \in \mathbf{R}$; i_0, \dots, i_{2^w-1} denote the standard generators of \mathcal{A}_w . Suppose that a foliation of \mathcal{A}_w by paths γ^α is so that

$$(6) \quad K(t, s) = K(\gamma^\alpha(\tau + p^\alpha \omega^\alpha), \gamma^\alpha(\kappa + p^\alpha \omega^\alpha)) = K(\gamma^\alpha(\tau), \gamma^\alpha(\kappa)), \text{ also } \phi_j(\gamma^\alpha(\kappa + p^\alpha \omega^\alpha)) = \phi_j(\gamma^\alpha(\kappa))$$

for μ almost all $t = \gamma^\alpha(\tau)$ and $s = \gamma^\alpha(\kappa)$ in \mathcal{A}_w for each α with $\omega^\alpha > 0$ and all integers p^α . Let $S(\omega)$ denote the shift operator

$$(7) \quad S(\nu)x(t) = x(t + \nu)$$

on a Cayley-Dickson number $\nu \in \mathcal{A}_w$, where $t \in \mathcal{A}_w$.

For example, a foliation $\{\gamma^\alpha : \alpha\}$ of \mathcal{A}_w may be done by straight lines parallel to $\mathbf{R}i_0$ indexed by $\alpha \in \Lambda = \{z \in \mathcal{A}_w : \operatorname{Re}(z) = 0\}$.

2. Definition. An operator $x \in L_q(L^p(\mathcal{A}_w, X), L^{p'}(\mathcal{A}_w, X))$ so that $(xf)(t) = x(t)f(t)$ and $x(t) \in L_q(X)$ for every $t \in \mathcal{A}_w$ we shall call periodic, if

$$(1) \quad S(\omega_j i_j)x(t)f(t) = x(t)S(\omega_j i_j)f(t)$$

for each $t \in \mathcal{A}_w$ and $f \in L^p(\mathcal{A}_w, X)$ and every j , where X is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v with $2 \leq v$. A set $\{\omega_j : j = 0, \dots, 2^w - 1\}$ will be called a net of periodic values. If for x exists a set of positive minimal periodic values, then it will be call a set of periods.

3. Definition. Let Y be a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , where $2 \leq v$. Put $l_\infty(\mathbf{Z}, Y) := \{x : x : \mathbf{Z} \rightarrow Y, \|x\|_\infty := \sup_{k \in \mathbf{Z}} \|x(k)\|_Y < \infty\}$ and $l_p(\mathbf{Z}, Y) := \{x : x : \mathbf{Z} \rightarrow Y, \|x\|_p := [\sum_{k \in \mathbf{Z}} \|x(k)\|_Y^p]^{1/p} < \infty\}$ to be the Banach spaces of norm $\|\cdot\|_p$ bounded sequences and with values in Y , where \mathbf{Z} denotes the ring of all integers, $1 \leq p < \infty$.

A sequence $\{x_n : n \in \mathbf{N}\} \subset l_\infty(\mathbf{Z}, Y)$ is called c -convergent to an element

$x \in l_\infty(\mathbf{Z}, Y)$, if for each integer $k \in \mathbf{Z}$ the limit is zero:

$$\lim_{n \rightarrow \infty} \|x_n(k) - x(k)\| = 0.$$

4. Definition. An operator $B \in L_q(l_\infty(\mathbf{Z}, Y))$ will be called c -continuous, if an image $\{Bx_n : n \in \mathbf{N}\}$ of each c -convergent sequence $\{x_n : n \in \mathbf{N}\}$ is a c -convergent sequence, where $L_q(X)$ is written shortly for $L_q(X, X)$, in particular for $X = l_\infty(\mathbf{Z}, Y)$, Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. The family of all c -continuous operators will be denoted by $L_q^c(l_\infty(\mathbf{Z}, Y))$.

5. Lemma. *The family $L_q^c(l_\infty(\mathbf{Z}, Y))$ from Definition 4 is a closed subalgebra over the Cayley-Dickson algebra \mathcal{A}_v in $L_q(l_\infty(\mathbf{Z}, Y))$ relative to the operator norm topology.*

Proof. Evidently, the unit operator I is c -continuous. Definitions 3 and 4 imply that $L_q^c(l_\infty(\mathbf{Z}, Y))$ is a subalgebra in $L_q(l_\infty(\mathbf{Z}, Y))$. Take an arbitrary sequence B_n of c -continuous operators converging to an operator $B \in L_q(l_\infty(\mathbf{Z}, Y))$ relative to the operator norm. Let x_n be an arbitrary c -converging sequence to x in $l_\infty(\mathbf{Z}, Y)$. From the c -continuity of an operator B_n it follows that for each $\epsilon > 0$ and $j \in \mathbf{Z}$ there exists a natural number m so that $\|(B_n(x_k - x))(j)\| < \epsilon/2$ for each $k > m$. On the other hand, the triangle inequality gives:

$$\begin{aligned} \|(B(x_k - x))(j)\| &\leq \|((B_n - B)(x_k - x))(j)\| + \|(B_n(x_k - x))(j)\| \\ &\leq \|((B_n - B)(x_k - x))(j)\| + \epsilon/2. \end{aligned}$$

The sequence B_n is norm convergent, hence there exists a natural number $l \in \mathbf{N}$ such that $\|(B(x_k - x))(j)\| < \epsilon$ for each $k > l$, since the limit $\lim_{k \rightarrow \infty} x_k(j) = x(j)$ exists for each j with $x \in l_\infty(\mathbf{Z}, Y)$ and $\sup_k \|(x_k - x)(j)\|_Y < \infty$. Thus the sequence Bx_k is c -convergent:

$$\lim_k Bx_k(j) = Bx(j) \text{ for each } j,$$

hence this operator B is c -continuous.

6. Definitions. Let $e_k \in l_p(\mathbf{Z}, \mathcal{A}_v)$ be basic elements so that $e_k(j) = \delta_{k,j}$, where $\delta_{k,j} = 0$ for each $j \neq k \in \mathbf{Z}$, while $\delta_{j,j} = 1$ for every $j \in \mathbf{Z}$. For an operator $B \in L_q(l_p(\mathbf{Z}, Y))$ with $1 \leq p \leq \infty$ and a Banach space Y over the Cayley-Dickson algebra \mathcal{A}_v let

$$(1) B_{j,k}y := (Be_ky)(j) \in Y$$

for each vector $y \in Y$, where $e_ky \in l_p(\mathbf{Z}, Y)$ with $(e_ky)(j) = \delta_{k,j}y$ for each j .

This set of operators $\{B_{j,k} : j, k \in \mathbf{Z}\}$ is called a matrix of an operator B .

An \mathcal{A}_v Banach subspace in $l_\infty(\mathbf{Z}, Y)$ of all two-sided sequences converging to zero $\lim_{|k| \rightarrow \infty} x(k) = 0$ is denoted by $c_0(\mathbf{Z}, Y)$.

7. Lemma. *Suppose that $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is a c -continuous non-zero operator or $B \in L_q(l_p(\mathbf{Z}, Y))$, where $1 \leq p < \infty$, Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then its matrix is non-zero and bounded. Moreover, $Bc_0(\mathbf{Z}, Y) \subset c_0(\mathbf{Z}, Y)$ for each $B \in L_q(l_\infty(\mathbf{Z}, Y))$.*

Proof. For each $B \in L_q(l_p(\mathbf{Z}, Y))$ we have the estimate:

$$\|B_{j,k}y\|_Y = \|(Be_ky)(j)\|_Y \leq \|Be_ky\|_p \leq$$

$$\|B\|_{L_q(l_p(\mathbf{Z}, Y))} \|e_ky\|_p = \|B\|_{L_q(l_p(\mathbf{Z}, Y))} \|y\|_Y$$

for each integers $j, k \in \mathbf{Z}$ and every vector $y \in Y$, hence $\sup_{j,k} \|B_{j,k}\| \leq \|B\|_{L_q(l_p(\mathbf{Z}, Y))}$. Particularly, for $B \in L_q(l_\infty(\mathbf{Z}, Y))$ and $x \in c_0(\mathbf{Z}, Y)$ this implies that

$$\|(B(x - \sum_{|k| \leq n} e_kx(k)))(j)\| \leq \|B\|_{L_q(l_\infty(\mathbf{Z}, Y))} \|(x - \sum_{|k| \leq n} e_kx(k))(j)\|_Y.$$

But for each $\epsilon > 0$ the set $\{k : \|x(k)\|_Y > \epsilon\}$ is finite, since $x \in c_0(\mathbf{Z}, Y)$, consequently,

$$\lim_{n \rightarrow \infty} \|x - \sum_{|k| \leq n} e_kx(k)\|_Y = 0$$

and hence $Bx \in c_0(\mathbf{Z}, Y)$.

Suppose that $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ has a zero matrix $\{B_{j,k} : j, k \in \mathbf{Z}\}$. Take an arbitrary vector $x \in l_\infty(\mathbf{Z}, Y)$ and a sequence $\{x_n : n \in \mathbf{Z}\} \subset l_\infty(\mathbf{Z}, Y) \cap c_0(\mathbf{Z}, Y)$ so that it c -converges to x . In the Banach space $c_0(\mathbf{Z}, Y)$ with norm $\|x_n\|_\infty$ the set of vectors $\{e_ky : y \in Y\}$ is everywhere dense. Since $\{B_{j,k} : j, k \in \mathbf{Z}\} = 0$, the restriction $B|_{(c_0(\mathbf{Z}, Y))}$ is zero and $Bx_n = 0$ for each n . Thus the sequence Bx_n does not converge to Bx . This produces the contradiction. Therefore, the matrix $\{B_{j,k} : j, k \in \mathbf{Z}\}$ is non-zero.

In the space $l_p(\mathbf{Z}, Y)$ with $1 \leq p < \infty$ a subset of finite two-sided sequences in Y is dense, hence a non-zero operator $B \in L_q(l_p(\mathbf{Z}, Y))$ has a non-zero matrix $\{B_{j,k} : j, k \in \mathbf{Z}\}$.

8. Definition. Let Λ be a Banach algebra with unit over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. A subalgebra Ξ over \mathcal{A}_v with unit is called saturated if each element of Ξ invertible in Λ is invertible in Ξ as well.

9. Remark. Henceforth in this section it will be denoted for short by $l(\mathbf{Z}, Y)$ each of the Banach spaces $c_0(\mathbf{Z}, Y)$ and $l_p(\mathbf{Z}, Y)$ with $p \in [1, \infty]$ if something other will not specified. Henceforward operators from $L_q^c(l_\infty(\mathbf{Z}, Y))$ will be considered.

Let $g \in l_\infty(\mathbf{Z}, \mathcal{A}_v)$, then an operator $D(g)$ is defined by the formula:

$$(1) \quad (D(g)x)(k) := g(k)x(k)$$

for each $x \in l_\infty(\mathbf{Z}, Y)$. The family of all such operators is denoted by \mathcal{E} . The family of all operators $A \in L_q^c(l_\infty(\mathbf{Z}, Y))$ quasi-commuting with each $D(g) \in \mathcal{E}$ is denoted by $L_q^s(l_\infty(\mathbf{Z}, Y))$, that is

$$(2) \quad {}^j A {}^k D(g) = (-1)^{\kappa(j,k)} {}^k D(g) {}^j A$$

for each $j, k = 0, 1, 2, \dots$ (see §§II.2.1 [28], 2.5 and 2.23 [29]).

Evidently, this subalgebra $L_q^s(l_\infty(\mathbf{Z}, Y))$ is saturated in the operator algebra $L_q^c(l_\infty(\mathbf{Z}, Y))$ with unit operator I as the unit of these algebras.

10. Definition. Take a Cayley-Dickson number of absolute value one $M \in \mathbf{S}_v := \{z : z \in \mathcal{A}_v, |z| = 1\}$ and put $\mathcal{M} = \{M^n : n \in \mathbf{Z}\} \in l_\infty(\mathbf{Z}, \mathcal{A}_v)$, where $2 \leq v$.

An operator $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is called diagonal if it quasi-commutes with $D(\mathcal{M}) \in \mathcal{E}$ for each $M \in \mathbf{S}_v$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. The family of all diagonal operators we denote by $L_q^d(l_\infty(\mathbf{Z}, Y))$.

11. Proposition. *The family of all diagonal operators $L_q^d(l_\infty(\mathbf{Z}, Y))$ is a saturated subalgebra in the algebra $L_q^c(l_\infty(\mathbf{Z}, Y))$.*

Proof. If $A, B \in L_q^d(l_\infty(\mathbf{Z}, Y))$, then equations 9(2) for A and B and $D(\mathcal{M})$ imply

$$(1) \quad ({}^j A + {}^j B) {}^k D(\mathcal{M}) = {}^j A {}^k D(\mathcal{M}) + {}^j B {}^k D(\mathcal{M}) = (-1)^{\kappa(j,k)} {}^k D(\mathcal{M}) ({}^j A + {}^j B)$$

for each $j, k = 0, 1, 2, \dots$ and every $M \in \mathbf{S}_v$ due to distributivity of the operator multiplication (see also §3.3.1). If $z \in \mathcal{A}_v$, then zA has components:

$$(2) \quad {}^j(zA) = \sum_{s,l; i_s i_l = i_j} (z_s i_s {}^l A + (-1)^{\kappa(s,l)} z_l i_l {}^s A)$$

for each j , where as usually $z = z_0 i_0 + z_1 i_1 + \dots$ with $z_0, z_1, \dots \in \mathbf{R}$. Therefore, zA quasi-commutes with each $D(\mathcal{M})$ for each $z \in \mathcal{A}_v$. Analogously it can be demonstrated for Az . Thus $L_q^d(l_\infty(\mathbf{Z}, Y))$ is an algebra.

An inverse $A^{-1} \in L_q^d(l_\infty(\mathbf{Z}, Y))$ of an operator $A \in L_q^d(l_\infty(\mathbf{Z}, Y))$ quasi-commuting with $D(\mathcal{M})$ also quasi-commutes with $D(\mathcal{M})$, since $D(\mathcal{M})$ is invertible and $(AD(\mathcal{M}))^{-1} = (D(\mathcal{M}))^{-1}A^{-1} = D(\mathcal{M}^*)A^{-1}$ and $(D(\mathcal{M})A)^{-1} = A^{-1}D(\mathcal{M}^*)$, $Ax = y$ implies $x = A^{-1}y$,

$$\begin{aligned} I &= (AD(\mathcal{M}))^{-1}(AD(\mathcal{M})) = \\ &= (AD(\mathcal{M}))^{-1} \sum_{j; i_s i_l = i_j} [{}^s A {}^l D(\mathcal{M}) + (-1)^{\kappa(s,l)} {}^l A {}^s D(\mathcal{M})] \\ &= (AD(\mathcal{M}))^{-1} \sum_{j; i_s i_l = i_j} [{}^s D(\mathcal{M}) {}^l A + (-1)^{\kappa(s,l)} {}^l D(\mathcal{M}) {}^s A] \\ &= \sum_{j; i_p i_q = i_j^*; i_s i_l = i_j} \{[{}^p D(\mathcal{M}^*) {}^q A^{-1} + (-1)^{\kappa(p,q)} {}^q D(\mathcal{M}^*) {}^p A^{-1}] \\ &\quad [{}^s D(\mathcal{M}) {}^l A + (-1)^{\kappa(s,l)} {}^l D(\mathcal{M}) {}^s A]\}, \end{aligned}$$

where \mathcal{M}^* corresponds to $M^* = \tilde{M}$.

12. Definition. A sequence $g \in l_\infty(\mathbf{Z}, \mathcal{A}_v)$ is called periodic of period $k \in \mathbf{N}$ if $g(n+k) = g(n)$ for each integer $n \in \mathbf{Z}$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$.

13. Lemma. Diagonal operators from $L_q^d(l_\infty(\mathbf{Z}, Y))$ quasi-commute with operators of multiplication on periodic sequences $g \in l_\infty(\mathbf{Z}, \mathcal{A}_v)$, where $2 \leq v$.

Proof. Let $A \in L_q^d(l_\infty(\mathbf{Z}, Y))$ be a diagonal operator and let $g \in l_\infty(\mathbf{Z}, \mathcal{A}_v)$ be a periodic sequence of period $k \in \mathbf{N}$. Put $\theta_j := \exp(2\pi i j/k)$, where $j = 0, 1, \dots, k-1$, \mathbf{i} is an additional purely imaginary generator so that $\mathbf{i}^2 = -1$, $\mathbf{i} i_l = i_l \mathbf{i}$ for each $l \geq 0$.

A minimal real algebra with basis of generators $i_0, i_1, \dots, i_{2^v-1}, \mathbf{i}, \mathbf{i} i_1, \dots, \mathbf{i} i_{2^v-1}$ and their relations as above is the complexification $(\mathcal{A}_v)_{\mathbf{C}_i}$ of the Cayley-Dickson algebra \mathcal{A}_v , where $\mathbf{C}_i = \mathbf{R} \oplus \mathbf{R} \mathbf{i}$. Then g can be presented in the form:

$$\begin{aligned} (1) \quad g(n) &= \sum_{j=0}^{k-1} c_j \theta_j^n, \text{ where} \\ (2) \quad c_j &= \frac{1}{k} \sum_{n=0}^{k-1} g(n) \tilde{\theta}_j^n \end{aligned}$$

are $(\mathcal{A}_v)_{\mathbf{C}_i}$ Fourier coefficients for each $j = 0, 1, \dots, k-1$. Indeed, $\mathbf{i}(\mathbf{i}x_j i_j) = -x_j i_j$ for each j and $x_j \in Y_j$, while $(a+b)c = ac+bc$ and $c(a+b) = ca+cb$ for each $c \in (\mathcal{A}_v)_{\mathbf{C}_i}$ and $a, b \in Y \oplus Y\mathbf{i}$, where we put $\mathbf{i}x = x\mathbf{i}$ for each $x \in Y$. Put $\|x+y\mathbf{i}\|^2 = \|x\|^2 + \|y\|^2$ for each $x, y \in Y$ and $x+y\mathbf{i} \in Y \oplus Y\mathbf{i}$. Therefore,

$$\begin{aligned} \sum_{j=0}^{k-1} c_j \theta_j^m &= \frac{1}{k} \sum_{j=0}^{k-1} \left[\sum_{n=0}^{k-1} \left(\sum_l g_l(n) i_l \right) \tilde{\theta}_j^n \right] \theta_j^m \\ &= \sum_l g_l(n) \frac{1}{k} \sum_{j=0}^{k-1} \sum_{n=0}^{k-1} i_l \theta_j^{m-n} = g(m), \end{aligned}$$

since the multiplication in the Cayley-Dickson algebra is distributive, where $g(m) = \sum_l g_l(m) i_l$ with $g_l(m) \in \mathbf{R}$ for each $l = 0, 1, 2, \dots$,

$$\frac{1}{k} \sum_{j=0}^{k-1} \exp(2\pi j \mathbf{i}(m-n)/k) = \delta_{n,m}.$$

Therefore, the diagonal operator has the decomposition

$$(D(g)x)(n) = g(n)x(n) = \left(\sum_{j=0}^{k-1} (c_j I) D(\hat{\theta}_j) \right) x(n)$$

that is

$$(3) \quad D(g) = \sum_{j=0}^{k-1} (c_j I) D(\hat{\theta}_j)$$

again due to distributivity of the multiplication in the Cayley-Dickson algebra \mathcal{A}_v (see §2.5 [29]). On the other hand, A and $D(\hat{\theta}_j)$ quasi-commute:

$${}^j A {}^k D(\hat{\theta}_j) = (-1)^{\kappa(j,k)} {}^k D(\hat{\theta}_j) {}^j A$$

for each j, k and from Proposition 11 it follows that A and $D(g)$ quasi-commute, where ${}^k D(\hat{\theta}_j)(n) \in \mathbf{C}_i i_k$ for each $n \in \mathbf{Z}$.

14. Corollary. *Diagonal operators from $L_q^d(l_\infty(\mathbf{Z}, Y))$ quasi-commute with multiplication operators $D(g)$ on periodic sequences $g \in l_\infty(\mathbf{Z}, \mathcal{A}_v)$, where $2 \leq v$.*

15. Definition. If $x \in l_p(\mathbf{Z}, Y)$, where $p \in [1, \infty]$, Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$, then its support is $\text{supp } x := \{n : n \in \mathbf{Z}; x(n) \neq 0\}$.

16. Lemma. *Let $B \in L_q^d(l_\infty(\mathbf{Z}, Y))$ be a diagonal operator and let $x \in l_\infty(\mathbf{Z}, Y)$ be with finite support, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then $\text{supp } Bx \subseteq \text{supp } x$.*

Proof. If the support of x is finite, then there exists a natural number $N \in \mathbf{N}$ so that $\text{supp } x \subseteq [-N, N]$. Consider natural numbers $m > N$ and $n \in [-N, N]$. Put $g(n + 2mk) = 1$ if $n \in \text{supp } x$, while $g(n + 2mk) = 0$ for $n \in [-N, N] \setminus \text{supp } x$, where $k \in \mathbf{Z}$, hence $g \in l_\infty(\mathbf{Z}, \mathcal{A}_v)$ is of period $2m$. Therefore, $D(g)x = x$ by the construction of g . In view of Corollary 14 $G(g)$ and B quasi-commute. On the other hand, $g(l) = 0$ for each $l \in [-m, m] \setminus \text{supp } x$, consequently, $(Bx)(l) = 0$, since $Bx = BD(g)x$. Thus $(Bx)(l) = 0$ for each $l \notin \text{supp } x$, since $m > N$ is arbitrary.

17. Theorem. *The algebras $L_q^s(l_\infty(\mathbf{Z}, Y))$ and $L_q^d(l_\infty(\mathbf{Z}, Y))$ over the Cayley-Dickson algebra \mathcal{A}_v coincide, where $2 \leq v$.*

Proof. From the definitions it follows that the algebra $L_q^s(l_\infty(\mathbf{Z}, Y))$ is a subalgebra of the algebra $L_q^d(l_\infty(\mathbf{Z}, Y))$ of diagonal continuous operators. Therefore, it remains to prove the inclusion $L_q^d(l_\infty(\mathbf{Z}, Y)) \subseteq L_q^s(l_\infty(\mathbf{Z}, Y))$. Consider arbitrary continuous diagonal operator $B \in L_q^d(l_\infty(\mathbf{Z}, Y))$ and an element $g \in l_\infty(\mathbf{Z}, \mathcal{A}_v)$. Take $x \in l_\infty(\mathbf{Z}, Y)$ so that $x_n = 0$ for each $|n| > N$, where N is a natural number. We have ${}^j D(g) {}^k Bx = (-1)^{\kappa(j,k)} {}^k B {}^j D(g)x$ for each j, k . Extend the sequence g periodically to h so that $h(-N) = g(N + 1)$ and $h(m) = g(m)$ for each $m \in [-N, N]$, consequently, $(D(g)x)(m) = (D(h)x)(m)$ for each $m \in [-N, N]$. In view of Corollary 14

$$\begin{aligned} ({}^j B {}^k D(g)x)(m) &= ({}^j B {}^k D(h)x)(m) = (-1)^{\kappa(j,k)} ({}^k D(h) {}^j Bx)(m) \\ &= (-1)^{\kappa(j,k)} ({}^k D(g) {}^j Bx)(m) \end{aligned}$$

for each j, k and $m \in [-N, N]$. Applying Lemma 16 for each $|m| > N$ we get $(Bx)(m) = 0$ and hence

$$({}^j B {}^k D(g)x)(m) = (-1)^{\kappa(j,k)} ({}^k D(g) {}^j Bx)(m)$$

for each $j, k = 0, 1, 2, \dots$ and every integer number $m \in \mathbf{N}$. Thus for sequences with finite supports this theorem is accomplished. But a set of all sequences with finite support is dense in $c_0(\mathbf{Z}, Y)$ and in $l_p(\mathbf{Z}, Y)$ for every $1 \leq p < \infty$. Therefore, the statement of this theorem is valid on these spaces. In accordance with Lemma 7 the conjecture spreads on a c -continuous operator B from $c_0(\mathbf{Z}, Y)$ on the entire Banach space $l_\infty(\mathbf{Z}, Y)$.

18. Definition. Let $g = (G^m : m \in \mathbf{Z})$ be a sequence belonging to the Banach space $l_\infty(\mathbf{Z}, L_q(Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. An operator $D(g) \in L_q^c(l_\infty(\mathbf{Z}, Y))$ will be defined by the formula: $(D(g)x)(m) = G^m x(m)$ for each integer number m . A set of all (left) multiplication operators on bounded operator valued sequences forms an algebra over the Cayley-Dickson algebra \mathcal{A}_v , which will be denoted by $L_q^b(l_\infty(\mathbf{Z}, Y))$.

An operator $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is called a (k, n) ribbon operator with $k \in \mathbf{N}$ and $n \in \mathbf{Z}$ if $B_{s,l} = 0_Y$ is the zero operator from $L_q(Y)$ for each $|s - l + n| \geq k$, where $s, l \in \mathbf{Z}$. Their family is denoted by $L_q^{(k,n)}(l_\infty(\mathbf{Z}, Y))$. A $(k, 0)$ ribbon operator is called k -ribbon (single ribbon for $k = 1$).

Two propositions follow immediately from the latter definition.

19. Proposition. *An operator $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is $(1, 0)$ ribbon if and only if it is a multiplication operator on operator valued sequence.*

20. Proposition. *The algebra $L_q^{(k,0)}(l_\infty(\mathbf{Z}, Y))$ is the saturated subalgebra of the algebra $L_q^c(l_\infty(\mathbf{Z}, Y))$ over the Cayley-Dickson algebra \mathcal{A}_v .*

21. Theorem. *Let $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. An operator B is $(1, 0)$ ribbon if and only if B is diagonal.*

Proof. Suppose that B is a diagonal operator. Take a vector $x \in Y$ and an element $y^s = e_s x \in l_\infty(\mathbf{Z}, Y)$, where $y^s(k) = \delta_{s,k} x$, hence $B_{m,s}(By^s)(m) \in Y$, where $s, m \in \mathbf{Z}$, $B_{s,m}$ are elements of the matrix of B . For $M \in \mathbf{S}_v$ matrix elements of the operator $D(\mathcal{M}^*)BD(\mathcal{M})$ are prescribed by the formula:

$$(D(\mathcal{M}^*)BD(\mathcal{M})y^s)(m) = M^{-m}(BM^s y^s)(m)$$

for each $s, m \in \mathbf{Z}$.

Take any purely imaginary generator i_p of the Cayley-Dickson algebra and put $M = i_p$ with $p \geq 1$. As an operator B is diagonal, the equalities follow:

$$({}^k B y^s)(m) = (-1)^{\kappa(p,k)\eta(s)} i_p^{s-m} ({}^k B y^s)(m)$$

for each m, s , where $\eta(s) = 0$ for s even, while $\eta(s) = 1$ for s odd. This implies that $B_{m,s} = 0_Y$ for $s \neq m$ and $B_{m,s} = B_{m,m}$ for $m = s$. Thus the operator B is $(1, 0)$ ribbon.

The inverse conjecture follows from Lemma 7 and Definition 18.

22. Corollary. *An operator $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is diagonal if and only if B is an operator of multiplication on a bounded operator valued sequence in $L_q(Y)$.*

23. Definition. An operator $B \in L_q(l_\infty(\mathbf{Z}, Y))$ is called uniformly c -continuous, if a mapping $\check{B} : \mathbf{S}^1 \rightarrow L_q(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i}))$ is continuous relative to the operator norm topology on $L_q(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i}))$ and a topology on $\mathbf{S}^1 := \{z : z \in \mathbf{C}_i; |z| = 1\}$ induced by the norm on the complex field \mathbf{C}_i , where $2 \leq v$, $\check{B}(M) := D(\mathcal{M})BD(\mathcal{M}^*)$ for each $M \in \mathbf{S}^1$, $D(\mathcal{M})$ is a diagonal (left) multiplication operator on a sequence $\mathcal{M}(k) = M^k$, Y is a Banach space over \mathcal{A}_v . The family of all uniformly c -continuous operators is denoted by $L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ and is supplied with the uniform operator norm topology

$$\|B\|_u := \sup_{M \in \mathbf{S}^1} \|\check{B}(M)\|.$$

The real field is the center of the Cayley-Dickson algebra \mathcal{A}_v with $v \geq 2$, hence the generator \mathbf{i} can be realized as the real matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If X and Y are two \mathcal{A}_v vector spaces and $B : X \rightarrow Y$ is a real homogeneous \mathcal{A}_v additive operator, then it has a natural extension $\mathbf{B} : X_{\mathbf{i}} \rightarrow Y_{\mathbf{i}}$ so that $\mathbf{B}(a + b\mathbf{i}) = (Ba) + (Bb)\mathbf{i}$ for each vectors $a, b \in X$, where $X_{\mathbf{i}}$ is obtained from X by extending the algebra \mathcal{A}_v to $(\mathcal{A}_v)_{\mathbf{C}_i}$ and $X_{\mathbf{i}}$ can be presented as the direct sum $X_{\mathbf{i}} = X \oplus X\mathbf{i}$ of two \mathcal{A}_v vector spaces X and $X\mathbf{i}$. It is convenient to denote \mathbf{B} also by B on $X_{\mathbf{i}}$.

24. Proposition. *The family $L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ is a normed algebra over the Cayley-Dickson algebra \mathcal{A}_v (see §23). If an operator $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is k -ribbon, then this operator B is uniformly c -continuous.*

Proof. In algebra $L_q(X)$ for a Banach space X over the Cayley-Dickson algebra \mathcal{A}_v the operator norm satisfies the inequality: $\|AB\| \leq \|A\|\|B\|$, particularly for $X = l_\infty(\mathbf{Z}, Y)$ or $X = l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i})$. Therefore, for each uniformly c -continuous operators A and B the inequality follows:

$$\begin{aligned} \|AB\|_u &= \sup_{M \in \mathbf{S}^1} \|D(\mathcal{M})ABD(\mathcal{M}^*)\| = \sup_{M \in \mathbf{S}^1} \|D(\mathcal{M})AD(\mathcal{M}^*)D(\mathcal{M})BD(\mathcal{M}^*)\| \\ &\leq \sup_{M \in \mathbf{S}^1} \|D(\mathcal{M})AD(\mathcal{M}^*)\| \sup_{M \in \mathbf{S}^1} \|D(\mathcal{M})BD(\mathcal{M}^*)\| = \|A\|_u \|B\|_u, \end{aligned}$$

since set theoretic composition of operators is associative and $D(\mathcal{M})D(\mathcal{M}^*) = I$ for each $M \in \mathbf{S}^1$ (see [32]). On the other hand, $\|B\|_u \geq \|B\|$ for each $B \in L_q^{uc}(l_\infty(\mathbf{Z}, Y))$, since $D(\mathcal{I}) = I$ and $1 \in \mathbf{S}^1$, where $\mathcal{I} = (\dots, 1, 1, \dots)$ corresponds to 1.

For a k ribbon operator B a finite sequence ${}_n B$ of single ribbon operators exists with $|n| \leq k$ so that

$$B = \sum_{n=-k}^k {}_n B S(n),$$

where $S(n)$ denotes a shift operator on n , $(S(n)g)(m) := g(m+n)$ for each $n, m \in \mathbf{Z}$ and $g \in l_\infty(\mathbf{Z}, Y)$. Therefore, the equality follows:

$$(\breve{B}(M)x)(j) = \sum_{n=-k}^k [(M^{j+n}I) {}_n B((M^*)^n I)x](j+n),$$

but $\|[(M^{j+n}I) {}_n B((M^*)^n I)y]\| \leq \|{}_n B y\|$ for each $y \in Y$ and $M \in \mathbf{S}^1$, consequently, $\|B\|_u \leq \sum_{n=-k}^k \|{}_n B\| < \infty$.

25. Lemma. *Let $B \in L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ be a uniformly c continuous operator and $(B_{s,p})$ be its matrix, where $s, p \in \mathbf{Z}$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then for each $M \in \mathbf{S}^1$ matrix elements of an operator $\breve{B}(M)$ have the form:*

$$\breve{B}(M)_{s,p} = (M^s I) B_{s,p} (M^{-p} I).$$

Proof. In accordance with Definition 6 the equalities are valid:

$$\begin{aligned} \breve{B}(M)_{s,p} &= (D(\mathcal{M})BD(\mathcal{M}^*))_{s,p}x = ((D(\mathcal{M})BD(\mathcal{M}^*))e_p x)(s) \\ &= ((M^s I)B(M^{-p}I)e_p x)(s) = ((M^s I)B_{s,p}(M^{-p}I)e_p x)(s) \end{aligned}$$

for all integer numbers $s, p \in \mathbf{Z}$ and for each vector $x \in Y$, since $M \in \mathbf{S}^1$ implies $|M|^2 = MM^* = M^*M = 1$ and hence $M^{-1} = M^*$.

26. Proposition. *Let $A, B \in L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ be two uniformly c -continuous operators, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then $(\breve{A}\breve{B})(M) = \breve{A}(M)\breve{B}(M)$ for each $M \in \mathbf{S}^1$.*

Proof. The algebra of operators relative to the set-theoretic composition is evidently associative (see also [32]), hence

$$(\breve{A}\breve{B})(M) = D(\mathcal{M})ABD(\mathcal{M}^*) = D(\mathcal{M})AD(\mathcal{M}^*)D(\mathcal{M})BD(\mathcal{M}^*)$$

$$= \check{A}(M)\check{B}(M) \text{ for each } M \in \mathbf{S}^1,$$

since $MM^* = M^*M = 1$ and hence $D(\mathcal{M})D(\mathcal{M}^*) = D(\mathcal{M}^*)D(\mathcal{M})$.

27. Proposition. *Let $B \in L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ be a uniformly c -continuous invertible operator, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then an operator $\check{B}(M)$ is invertible so that $\check{B}^{-1}(M) = (\check{B})^{-1}(M)$ for each $M \in \mathbf{S}^1$.*

Proof. Applying Proposition 26 with $A = B^{-1}$ one gets $(\check{B})^{-1}(M) = (D(\mathcal{M})BD(\mathcal{M}^*))^{-1} = D(\mathcal{M}^*)^{-1}B^{-1}D(\mathcal{M})^{-1} = D(\mathcal{M})B^{-1}D(\mathcal{M}^*) = \check{B}^{-1}(M)$.

28. Lemma. *Let $B \in L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ be a uniformly c -continuous operator, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then there exists an equivalent norm on Y relative to an initial one so that $\|B\| \geq \|\check{B}(M)\|$ for each $M \in \mathbf{S}^1$.*

Proof. The norm on the Cayley-Dickson algebra \mathcal{A}_v satisfies the inequality $|ab| \leq |a||b|$ for each $a, b \in \mathcal{A}_v$ with $2 \leq v$. Particularly, for $v \leq 3$ the norm on \mathcal{A}_v is multiplicative.

Two norms $\|*\|$ and $\|*\|'$ on a Banach space Y are called equivalent if two positive constants $0 < c_1 \leq c_2 < \infty$ exist so that $c_1\|x\| \leq \|x\|' \leq c_2\|x\|$ for each vector $x \in Y$. Then $\|ax\| \leq \|a\|\|x\|$ for each $a \in \mathcal{A}_v$ and $x \in Y$ up to a topological isomorphism of Banach spaces, i.e. up to an equivalence of norms on Y , since $\|tx_ji_j\| = |t|\|x_j\| = \|tx_j\|$ for each $x_j \in Y_j$ and $j = 0, 1, 2, \dots$. Indeed, the multiplication of vectors on numbers $\mathcal{A}_v \times Y \ni (a, x) \mapsto ax \in Y$ is continuous relative to norms on \mathcal{A}_v and Y . Therefore, $\|\check{B}(M)x\| = \|D(\mathcal{M})BD(\mathcal{M}^*)x\| \leq \|D(\mathcal{M})\|\|B\|\|D(\mathcal{M}^*)\|\|x\|$, consequently, $\|B\| \geq \|\check{B}(M)\|$ for each $M \in \mathbf{S}^1$, since $|M|^n = |M^n| = 1$ for each integer n and hence $\|D(\mathcal{M})\| = 1$.

29. Definition. Let $C_s(\mathbf{S}^1, L_q^{uc}(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i})))$ denote a Banach space of continuous bounded mappings from \mathbf{S}^1 into $L_q^{uc}(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i}))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$.

30. Corollary. *There exists an equivalent norm on a Banach space Y over the Cayley-Dickson algebra \mathcal{A}_v such that the mapping $F : L_q^{uc}(l_\infty(\mathbf{Z}, Y)) \rightarrow C_s(\mathbf{S}^1, L_q^{uc}(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i})))$ given by the formula $F(B)(M) = \check{B}(M)$ for each $M \in \mathbf{S}^1$ is \mathbf{R} linear and \mathcal{A}_v additive and isometric operator.*

Proof. This follows by combining Proposition 24 and Lemma 28.

31. Lemma. Let $B \in L_q^{uc}(l_\infty(\mathbf{Z}, Y))$. Then an operator valued mapping $\check{B} : \mathbf{S}^1 \rightarrow L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ has the Fourier series of the form:

$$(1) \quad \check{B} \sim \sum_{n=-\infty}^{\infty} M^n {}_n\check{B}$$

for each $M \in \mathbf{S}^1$, where

$$(2) \quad {}_n\check{B} = \frac{1}{2\pi} \int_0^{2\pi} e^{-nti} \check{B}(e^{ti}) dt$$

are Fourier coefficients. Moreover, each operator ${}_n\check{B}S(-n)$ is diagonal.

Proof. From the definition of the uniformly c -continuous operator it follows that the restriction of \check{B} on \mathbf{S}^1 is continuous, since a mapping $\check{B} : \mathbf{S}^1 \rightarrow L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ is continuous. The function e^{ti} has the period 2π . Therefore, integrals (2) exist for every n and a formal Fourier series (1) can be written. On the other hand, the algebra $\text{alg}_{\mathbf{R}}(i_k, i_l)$ is associative for every k and $l = 0, 1, 2, \dots$. Therefore, using the distributivity law in the algebra $(\mathcal{A}_v)_{\mathbf{C}_i}$ we deduce that

$$M^{\mp k} B_{k,l} M^{\pm l} \sum_p x_p i_p = M^{\mp k} B_{k,l} \sum_p M^{\pm l} x_p i_p,$$

where $M \in \mathbf{S}^1$, $x_p \in Y_p$ for each $p = 0, 1, 2, \dots$. The algebra $\text{alg}_{\mathbf{R}}(\mathbf{i}, i_p, i_k)$ is associative for each p, k , consequently, the inversion formula (1) is valid, since

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-nti} e^{mti} {}_m\check{B}x dt = \delta_{m,n} {}_m\check{B}x$$

for each vector $x \in Y$. Then one gets

$$\begin{aligned} D(\mathcal{M}) {}_n\check{B}S(-n)D(\mathcal{M}^*) &= D(\mathcal{M}) {}_n\check{B}D(\mathcal{M}^*)D(\mathcal{M})S(-n)D(\mathcal{M}^*) \\ &= (M^n I) {}_n\check{B}(M^{-n} I)(M^n I)S(-n)(M^{-n} I) = (M^n I) {}_n\check{B}S(-n)(M^{-n} I) \end{aligned}$$

for each $M \in \mathbf{S}^1$ and every integer n , since the product of diagonal operators is diagonal.

32. Theorem. The algebra $L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ is the saturated subalgebra in $L_q^c(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$.

Proof. The algebraic \mathbf{R} linear \mathcal{A}_v additive embedding $L_q^{uc}(l_\infty(\mathbf{Z}, Y)) \hookrightarrow L_q^c(l_\infty(\mathbf{Z}, Y))$ follows from the definitions. If a uniformly c -continuous operator B on $l_\infty(\mathbf{Z}, Y)$ is invertible in the algebra $L_q^c(l_\infty(\mathbf{Z}, Y))$, then the mapping

\check{B}^{-1} is continuous from \mathbf{S}^1 into $L_q^c(l_\infty(\mathbf{Z}, Y))$ due to Proposition 27. Thus $B^{-1} \in L_q^{uc}(l_\infty(\mathbf{Z}, Y))$.

33. Definition. An operator $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is called periodic of period n on the Banach space $l_\infty(\mathbf{Z}, Y)$ over the Cayley-Dickson algebra \mathcal{A}_v with $2 \leq v$ if $S(n)B = BS(n)$, where n is a natural number, $(S(n)x)(k) = x(n+k)$ for each vector $x \in l_\infty(\mathbf{Z}, Y)$ and every integer k . A set of n periodic operators will be denoted by $L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$.

34. Proposition. A set $L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ of n periodic operators on $l_\infty(\mathbf{Z}, Y)$ is a closed saturated subalgebra in the algebra $L_q^c(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$.

Proof. If operators $A, B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ commute with $S(n)$, then

$$\begin{aligned} [(\alpha I)A + B(\beta I)]S(n) &= (\alpha I)S(n)A + BS(n)(\beta I) \\ &= S(n)(\alpha I)A + S(n)B(\beta I) = S(n)[(\alpha I)A + B(\beta I)] \end{aligned}$$

for each Cayley-Dickson numbers $\alpha, \beta \in \mathcal{A}_v$ and

$$ABS(n) = AS(n)B = S(n)AB.$$

Thus $L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ is an algebra over the Cayley-Dickson algebra \mathcal{A}_v .

From Definition 33 the algebraic \mathbf{R} linear \mathcal{A}_v additive embedding $L_q^{n,per}(l_\infty(\mathbf{Z}, Y)) \hookrightarrow L_q^c(l_\infty(\mathbf{Z}, Y))$ follows.

The relation $S(n)B - BS(n) = 0$ defines a closed subset in $L_q^c(l_\infty(\mathbf{Z}, Y))$, since $S(n)$ is the bounded continuous operator on $l_\infty(\mathbf{Z}, Y)$ and the mapping $f(B) := S(n)B - BS(n)$ is continuous from $L_q^c(l_\infty(\mathbf{Z}, Y))$ into itself $L_q^c(l_\infty(\mathbf{Z}, Y))$. Thus $L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ is the closed subalgebra.

If an n periodic operator $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ is invertible in $L_q^c(l_\infty(\mathbf{Z}, Y))$, then $B^{-1}S(n) = S(n)B^{-1}$, since $BS(n) = S(n)B \Leftrightarrow S(n) = B^{-1}S(n)B \Leftrightarrow S(n)B^{-1} = B^{-1}S(n)$. Thus $B^{-1} \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ and hence the subalgebra $L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ is saturated in $L_q^c(l_\infty(\mathbf{Z}, Y))$.

35. Lemma. An operator $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is n periodic if and only if its matrix satisfies the condition $B_{k+n, l+n} = B_{k, l}$ for each integers k and l , where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$.

Proof. Suppose that $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ and $(B_{k, l})$ is its matrix. Then $B_{k, l}x = (Be_lx)(k) = (S(-n)BS(n)e_lx)(k) = (S(-n)Be_{l+n}x)(k) = (Be_{l+n}x)(k+n) = B_{k+n, l+n}x$ for each vector $x \in Y$ and integers k and l .

Vise versa if $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$ is a c -continuous operator, then it has

a matrix $(B_{k,l})$ by Lemma 7. Then the condition $B_{k+n,l+n} = B_{k,l}$ for each integers k and l implies $B_{k+n,l+n}x = (Be_{l+n}x)(k+n) = (S(-n)Be_{l+n}x)(k) = (S(-n)BS(n)e_lx)(k) = (Be_lx)(k)$, consequently, $S(-n)BS(n) = B$ and hence $BS(n) = S(n)B$. Thus the operator B is n -periodic.

36. Corollary. *Suppose that B is a c -continuous operator $B \in L_q^c(l_\infty(\mathbf{Z}, Y))$. Then B is n -periodic and diagonal if and only if it is a (left) multiplication operator on a stationary n -periodic sequence in $L_q(Y)$.*

Proof. This follows from Theorem 17 and Lemma 35.

37. Definition. A function $\hat{B} : \mathbf{S}^1 \rightarrow L_q(Y \oplus Y\mathbf{i})$ prescribed by the formula $\hat{B}(M)x := \bigoplus_{j=0}^{n-1} B(D(\mathcal{M})x)(j)$ will be called the Fourier transform of an n -periodic operator $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. We put

$$\|\hat{B}(M)x\| := \max_{j=0}^{n-1} \|B(D(\mathcal{M})x)(j)\|.$$

By $C_s(\mathbf{S}^1, (L_q(Y \oplus Y\mathbf{i}))^n)$ will be denoted the Banach space of all bounded continuous mappings $G : \mathbf{S}^1 \rightarrow L_q(Y \oplus Y\mathbf{i})$ supplied with the norm

$$\|G\| := \sup_{M \in \mathbf{S}^1} \max_{j=0}^{n-1} \|_j G(D(\mathcal{M}))\|,$$

where $\|A\|$ denotes a norm of an operator $A \in L_q(Y)$, Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$, $G = \bigoplus_{j=0}^{n-1} {}_j G$ with ${}_j G \in L_q(Y \oplus Y\mathbf{i})$ for every j .

38. Lemma. *Let $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ be a periodic operator, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then $\hat{B} \in C_s(\mathbf{S}^1, (L_q(Y \oplus Y\mathbf{i}))^n)$.*

Proof. A uniform space $C_s(\mathbf{S}^1, L_q(Y \oplus Y\mathbf{i}))$ is complete for a Banach space Y over the Cayley-Dickson algebra \mathcal{A}_v . Take an arbitrary vector $x \in Y$ and a complex number $K \in \mathbf{S}^1$ and a sequence ${}_k M \in \mathbf{S}^1$ converging to K . The Banach spaces $(L_q(Y \oplus Y\mathbf{i}))^n$ and $\bigoplus_{j=0}^{n-1} L_q(Y \oplus Y\mathbf{i})$ are isometrically isomorphic when supplied with the corresponding norms, since n is a natural number, where

$$(1) \|A\| = \sup_{0 \leq j \leq n-1} \|_j A\|$$

for each $A = ({}_0 A, \dots, {}_{n-1} A) \in \bigoplus_{j=0}^{n-1} L_q(Y \oplus Y\mathbf{i})$, also

$$(2) \|x\| = \sup_{0 \leq j \leq n-1} \|_j x\|$$

for each $x = ({}_0x, \dots, {}_{n-1}x) \in \bigoplus_{j=0}^{n-1} (Y \oplus Y\mathbf{i})$. Then a sequence $\{B(D({}_k\mathcal{M})x) : k \in \mathbf{N}\}$ c -converges to $B(D(\mathcal{K})x)$, since an operator B is c -continuous. By Definitions 3 and 37 this means that the limit exists

$$\lim_{k \rightarrow \infty} \|\hat{B}({}_k\mathcal{M})x - \hat{B}(\mathcal{K})x\| = \lim_{k \rightarrow \infty} \max_{j=0}^{n-1} \|B(D({}_k\mathcal{M})x)(j) - B(D(\mathcal{K})x)(j)\| = 0$$

and hence $\|\mathbf{F}\| \leq 1$, where \mathbf{F} denotes the Fourier transform operator on $L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ with values in $C_s(\mathbf{S}^1, (L_q(Y \oplus Y\mathbf{i}))^n)$.

39. Corollary. *If a sequence $\{B_p : p \in \mathbf{N}\}$ of n -periodic operators converges to an n -periodic operator B relative to the norm on $L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$, then a sequence of their Fourier transforms $\mathbf{F}(B_p)$ converges to $\mathbf{F}(B)$ in $C_s(\mathbf{S}^1, (L_q(Y \oplus Y\mathbf{i}))^n)$.*

40. Corollary. *If B is an n -periodic operator and $\mathbf{F}(B) = \hat{B}$ its Fourier transform, then $\|B\| \geq \sup_{M \in \mathbf{S}^1} \|\hat{B}(M)\|$.*

41. Notation. A family of all n -periodic operators $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ such that its Fourier transform $\mathbf{F}(B) = \hat{B}$ has an absolutely converging Fourier series

$$\hat{B}(M) = \sum_{k=-\infty}^{\infty} \bigoplus_{j=0}^{n-1} M^{(k-1)n+j} {}_{(k-1)n+j}B,$$

i.e. $\sum_{k=-\infty}^{\infty} \max_{j=0}^{n-1} \|{}_{(k-1)n+j}B\| < \infty$, will be denoted by $L_q^{n,1}(l_\infty(\mathbf{Z}, Y))$.

42. Lemma. *Let $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ be an n -periodic operator and its Fourier transform $\mathbf{F}(B) = \hat{B}$ has the form:*

$$\hat{B}(M) = \sum_{l=-\infty}^{\infty} M^l {}_lB,$$

where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$.

Then

$$(1) \quad B = \sum_{k=-\infty}^{\infty} {}_k\bar{B}S(kn),$$

where ${}_k\bar{B} = \bigoplus_{j=0}^{n-1} {}_{(k-1)n+j}B \in \bigoplus_{j=0}^{n-1} L_q^c(l_\infty(\mathbf{Z}, Y))$ is an operator of (left) multiplication on stationary operator valued sequence ${}_k\bar{B} \in \bigoplus_{j=0}^{n-1} L_q(Y)$, moreover,

$$(2) \quad \|{}_k\bar{B}\| = \sup_{0 \leq j \leq n-1} \|{}_{(k-1)n+j}B\| \text{ for each } k.$$

Proof. In view of Lemma 7 the Banach spaces $\bigoplus_{j=0}^{n-1} L_q^c(l_\infty(\mathbf{Z}, Y))$ and $L_q^c(l_\infty(\mathbf{Z}, Y))$ are isometrically isomorphic, that follows from using the block

form of matrices $(B_{(k-1)n+j, (m-1)n+l})$ of operators B , where $k, m \in \mathbf{Z}$ and $j, l = 0, \dots, n-1$, $n \geq 1$. From Lemma 5 it follows that an operator $\sum_{k=-\infty}^{\infty} (k-1)n+j BS(kn)$ is c -continuous for each j , consequently, $\sum_{k=-\infty}^{\infty} k \bar{B}S(kn)$ is also c -continuous. A matrix of the operator B coincides with that of $\sum_{k=-\infty}^{\infty} k \bar{B}S(kn)$ by Lemma 35 and Definition 37. Therefore, Formula (1) is satisfied in accordance with Lemma 7. The natural isometric embedding $Y \hookrightarrow Y \oplus Y\mathbf{i}$ induces isometric embeddings $L_q(Y) \hookrightarrow L_q(Y \oplus Y\mathbf{i})$ and $L_q(l_\infty(\mathbf{Z}, Y)) \hookrightarrow L_q(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i}))$ of normed spaces over the Cayley-Dickson algebra \mathcal{A}_v .

From the definition of the operator norms on $L_q(Y \oplus Y\mathbf{i})$ and $\bigoplus_{j=0}^{n-1} L_q(Y \oplus Y\mathbf{i})$ (see Formulas 38(1, 2)) and $L_q^c(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i}))$ Equality (2) follows, where

$$(3) \quad \|A\| = \sup_{0 \leq j \leq n-1} \|_j A\|$$

for each $A = ({}_0 A, \dots, {}_{n-1} A) \in \bigoplus_{j=0}^{n-1} L_q^c(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i}))$.

43. Corollary. *If $B, D \in L_q^{n, per}(l_\infty(\mathbf{Z}, Y))$, then*

$$\begin{aligned} (1) \quad (Bx)(l) &= \sum_{s=-\infty}^{\infty} {}_s Bx(s+l) \\ &= \sum_{s=-\infty}^{\infty} {}_{s-l} Bx(s) =: (b \star x)(l) \end{aligned}$$

and $(BD)(x) = b \star (d \star x)$ for each $x \in l_\infty(\mathbf{Z}, Y)$, where $b = \{{}_s B : s \in \mathbf{Z}\} \in l_1(\mathbf{Z}, L_q(Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v . Particularly, as $0 \leq v \leq 2$ the convolution is associative $b \star (d \star x) = (b \star d) \star x$.

Proof. The equalities follow

$$\begin{aligned} (Bx)(l) &= \sum_{k=-\infty}^{\infty} \sum_{j=0}^{n-1} {}_{(k-1)n+j} Bx((k-1)n+j+l) \\ &= \sum_{k=-\infty}^{\infty} \sum_{j=0}^{n-1} {}_{(k-1)n+j-l} Bx((k-1)n+j) \end{aligned}$$

from Lemma 42. Putting $s = (k-1)n+j$ one gets Formula (1).

44. Remark. If $n = 1$, the Fourier transform of an operator valued function $b : \mathbf{Z} \rightarrow L_q(Y)$ with $b \in l_1(\mathbf{Z}, L_q(Y))$ coincides with the Fourier series for a mapping \check{B} .

45. Proposition. *Let A and B be two operators in $L_q^{n, per}(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then*

$$(1) \quad \widehat{AB}(M) = \hat{A}(M)\hat{B}(M)$$

for each $M \in \mathbf{S}^1$.

Proof. With $M \in \mathbf{S}^1$ we infer that

$$(2) \quad {}_m(AB) = \sum_{p=0}^m {}_pA {}_{m-p}B$$

and this implies Formula (1), since $\widehat{AB}(M)x = A(D(\mathcal{M})B(D(\mathcal{M})x))$ for each $x \in Y$ and $M \in \mathbf{S}^1$, since $\mathbf{i}i_j = i_j\mathbf{i}$ for each j .

46. Proposition. *Let an operator $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ be n -periodic, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then an operator $\hat{B}(M)$ is invertible and $(\hat{B}(M))^{-1}x = \widehat{A}^{-1}(M)x$ for each $M \in \mathbf{S}^1$ and $x \in Y$.*

Proof. If $N \in \mathbf{S}^1$, then an algebra $\text{alg}_{\mathbf{R}}(N, i_s)$ is associative for each $s \geq 0$, since $N = N_0 + N_1\mathbf{i}$ with $N_0, N_1 \in \mathbf{R}$ and $\mathbf{i}i_s = i_s\mathbf{i}$ for each $s \geq 0$. If $M \in \mathbf{S}^1$ and $x \in Y$, one can take the algebra $\text{alg}_{\mathbf{R}}(M)$ which is either the real or complex field. Therefore, $B(D(\mathcal{M})B^{-1}(D(\mathcal{M})x)) = AA^{-1}x = x$ with $Ax = B(D(\mathcal{M})x)$ by Proposition 45.

47. Corollary. *Let $B, D \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ be n -periodic operators and let $(\hat{B}(M))^{-1}x = \hat{D}(M)x$ for each $M \in \mathbf{S}^1$ and $x \in Y$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then $D = B^{-1}$.*

Proof. This follows from Proposition 46, since the \mathbf{R} linear span $\text{span}_{\mathbf{R}}\{y = Mx : M \in \mathbf{S}^1, x \in X\}$ of such set of vectors is isomorphic with $X \oplus X\mathbf{i}$.

48. Lemma. *Let an n -periodic operator $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ be uniformly c -continuous, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then its Fourier transform \hat{B} is uniformly c -continous, $\hat{B} \in L_q^{uc}(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i}))$.*

Proof. From the conditions of this lemma it follows, that the mapping $\check{B} : \mathbf{S}^1 \ni M \mapsto D(\mathcal{M})BD(\mathcal{M}^*)$ is continuous from \mathbf{S}^1 into $L_q^{n,per}(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i}))$. Up to an \mathbf{R} -linear continuous algebraic automorphism of the Cayley-Dickson algebra \mathcal{A}_v and the corresponding automorphism of a Banach space Y , the Fourier series

$$(1) \quad \check{B}(M) \sim \sum_{k=-\infty}^{\infty} M^k {}_k\check{B}$$

exists by Lemma 31. This series converges to $\check{B}(M)$ by Cezaro, that is

$$(2) \quad \check{B}(M) = \lim_{m \rightarrow \infty} \sum_{k=-m}^m (1 - \frac{|k|}{m+1}) M^k {}_k\check{B},$$

since $(1 - \frac{|k|}{m+1}) \in \mathbf{R}$ while the real field is the center of the Cayley-Dickson algebra \mathcal{A}_v . Particularly, for $M = 1$ one has $M^k = 1$ and $\check{B}(1) = B$ (see also §20.2(743) [8]).

The operator B is n -periodic, so consider its Fourier transform and get

$$(3) \quad \hat{B}(M) = \lim_{m \rightarrow \infty} \sum_{k=-m}^m (1 - \frac{|k|}{m+1}) {}_k\hat{B}(M).$$

In view of Lemma 31 each operator ${}_k\check{B}S(-k)$ is diagonal. Since B is n -periodic, this implies that every operator $\bigoplus_{j=0}^{n-1} ({}_{(k-1)n+j}\check{B}S(-(k-1)n-j))$ is n -periodic as well. Therefore, ${}_k\hat{B}(M)x = ({}_k\check{B}(D(\mathcal{M})x))(0) = (M^k I) {}_kBx$ for each vector $x \in Y \oplus Y\mathbf{i}$, since $M^k(M^{-k}x_l i_l) = M^k(M^{-k}i_l x_l) = x_l i_l$ for each $l \geq 0$ and $x_l \in Y_l$, consequently, ${}_k\hat{B}(M) = (M^k I) {}_kB$ and hence

$$(4) \quad \hat{B}(M) = \lim_{m \rightarrow \infty} \sum_{k=-m}^m (1 - \frac{|k|}{m+1}) (M^k I) {}_kB,$$

where $({}_k\check{B})_{s,p} = {}_kB$ for each $s - p = k$, $s, p \in \mathbf{Z}$.

49. Notation. Let \mathbf{P} be a Banach algebra over the Cayley-Dickson algebra \mathcal{A}_v with $2 \leq v$. We denote by $F(\mathbf{S}^1, \mathbf{P})$ a Banach space of all continuous functions $f : \mathbf{S}^1 \rightarrow \mathbf{P}$ with absolutely converging Fourier series

$$(1) \quad f(M) = \sum_{k=-\infty}^{\infty} M^k {}_kf$$

relative to the norm:

$$(2) \quad \|f\| := \sum_{k=-\infty}^{\infty} \|{}_kf\|,$$

where ${}_kf \in \mathbf{P}$ for each $k \in \mathbf{Z}$.

50. Corollary. Let $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$. Then the following conditions are equivalent:

- (1) $\hat{B} \in F(\mathbf{S}^1, L_q(Y \oplus Y\mathbf{i}))$ and
- (2) $\check{B} \in F(\mathbf{S}^1, L_q^c(l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i})))$.

Proof. This follows from Lemma 48, since $\|{}_k\check{B}\| = \|{}_kB\|$ for each $k \in \mathbf{Z}$. Indeed, generally $\|{}_k\check{B}x\| = \|{}_kBx\|$, since $|ab| \leq |a||b|$ for each Cayley-Dickson numbers $a, b \in \mathcal{A}_v$ and $\|ax\| \leq |a|\|x\|$ for each $a \in \mathcal{A}_v$ and $x \in Y$ (see §I.2.1 [28]). In particular, if $x \in Y_0$ or $x \in Y_l$, then $\|ax\| = |a|\|x\|$.

51. Theorem. *The algebra $L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ is the subalgebra of the algebra $L_q^{uc}(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over the Cayley-Dickson algebra \mathcal{A}_v , $2 \leq v$.*

Proof. If $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$, then by Lemma 42 we have

$$B = \sum_{k=-\infty}^{\infty} {}_k\bar{B}S(kn),$$

where ${}_k\bar{B} = \bigoplus_{j=0}^{n-1} (k-1)n+j B \in \bigoplus_{j=0}^{n-1} L_q^c(l_\infty(\mathbf{Z}, Y))$ is an operator of (left) multiplication on stationary operator valued sequence ${}_k\tilde{B} \in \bigoplus_{j=0}^{n-1} L_q(Y)$, moreover,

$\|{}_k\bar{B}\| = \|{}_k\tilde{B}\|$ for each k . In view of Proposition 24 an 1-ribbon operator ${}_k\bar{B}$ is uniformly c -continuous. On the other hand, each shift operator $S(n)$ is uniformly c -continuous. The algebra $L_q^{uc}(l_\infty(\mathbf{Z}, Y))$ is complete as the uniform space. Therefore, the operator B is uniformly c -continuous.

3 Fourier transform on algebras and spectra

52. Definitions. Let \mathbf{G} be a quasi-group, i.e. a set with one binary operation (multiplication) so that

- (1) there exists a unit element e so that $eb = be = b$;
- (2) each element b has an inverse b^{-1} , i.e. $b^{-1}b = bb^{-1} = e$;
- (3) a multiplication is alternative $(aa)b = a(ab)$ and $b(aa) = (ba)a$ and
- (4) $a^{-1}(ab) = b$ and $(ba)a^{-1} = b$ for each $a, b \in \mathbf{G}$.

Let R_a be a Banach algebra over the real field \mathbf{R} for each $a \in \mathbf{G}$ such that R_a is isomorphic with R_b for all $a, b \in \mathbf{G}$. Put

$$(5) \quad R = \{B : B \in \bigoplus_{a \in \mathbf{G}} aR_a, \|B\| < \infty\},$$

$$\mathbf{G}_{\mathbf{R}} =: \{x : x \in \bigoplus_{a \in \mathbf{G}} a\mathbf{R}, |x| < \infty\}$$

is a quasi-group ring over the real field so that $a\beta = \beta a$ for each $a \in \mathbf{G}$ and $\beta \in \mathbf{R}$,

$$(6) \quad |x|^2 = \sum_{a \in \mathbf{G}} |x_a|^2 \text{ for } x = \sum_{a \in \mathbf{G}} x_a a$$

with $x_a \in \mathbf{R}$ for each $a \in \mathbf{G}$;

$$(7) \quad \|A\|^2 := \sum_{b \in \mathbf{G}} \|A_b\|^2$$

for $A = \sum_{b \in \mathbf{G}} A_b b$ with $A_b \in R_b$ for each $b \in \mathbf{G}$,

(8) $bA_a = A_a b$ for each $A_a \in R_a$ and $a, b \in \mathbf{G}$. Suppose that R_a

(9) contains a unit element I and that

(10) $\|I\| = 1$ and

(11) $\|AB\| \leq \|A\|\|B\|$ for each $A, B \in R_a$.

Denote by $L(R)$ the Banach space of all strongly integrable functions $f : \mathbf{R} \rightarrow R$ supplied with the norm

$$(12) \quad \|f\| := \int_{-\infty}^{\infty} \|f(t)\| dt < \infty.$$

Henceforward, we suppose that an algebra R

(13) is alternative $(AA)B = A(AB)$ and $B(AA) = (BA)A$ for each $A, B \in R$ and

(14) if A is left invertible, then also $A^{-1}(AB) = B$, if A is right invertible $(BA)A^{-1} = B$ for every $A, B \in R$.

The alternativity implies that \mathbf{G} and R are power-associative that is $b^m b^n = b^{n+m}$ and $B^m B^n = B^{n+m}$ for each $b \in \mathbf{G}$ and $B \in R$ and natural numbers n, m , where $b^n = b(b(\dots(bb)\dots))$ denotes the n -fold product, $b^0 = e$ for $b \neq 0$, $B^0 = I$ for $B \neq 0$.

We consider their complexifications $\mathbf{G}_{\mathbf{C}_i} := \mathbf{G}_{\mathbf{R}} \oplus \mathbf{G}_{\mathbf{R}} \mathbf{i}$ and $R_{\mathbf{C}_i} := R \oplus R \mathbf{i}$, where $\mathbf{C}_i = \mathbf{R} \oplus \mathbf{R} \mathbf{i}$ with $a \mathbf{i} = \mathbf{i} a \in \mathbf{G}_{\mathbf{C}_i}$,

(15) $|a + b \mathbf{i}|^2 = |a|^2 + |b|^2$ for each $a, b \in \mathbf{G}_{\mathbf{R}}$ and

(16) $\|A + B \mathbf{i}\|^2 = \|A\|^2 + \|B\|^2$ for every $A, B \in R$.

Analogously a Banach space $L(R_{\mathbf{C}_i})$ is defined with $f : \mathbf{R} \rightarrow R_{\mathbf{C}_i}$.

53. Lemma. *Let R be an algebra as in §52 and let an element $A \in R$ be of norm $\|A\| < 1$, then the series $C = I - A + A^2 - A^3 + \dots$ is absolutely convergent and*

$$(1) \quad C(I + A) = (I + A)C = I.$$

Proof. From Formulas 52(7, 11, 16) it follows that $\|A^n\| \leq \|A\|^n$ for each natural number n , consequently, the sequence of partial sums $S_n := I - A + A^2 - A^3 + \dots + (-1)^n A^n$ converges in R . A Banach algebra R is power-associative and this implies Formula (1).

54. Lemma. *Let R be an algebra as in §52 and let an element $A \in R$ have a left inverse Q . If $B \in R$ is an element such that $\|B\|\|Q\| < 1$, then*

$(A + B)$ has a left inverse C so that

$$(2) \quad C = Q(I - BQ + (BQ)^2 - (BQ)^3 + \dots).$$

Proof. A Banach algebra R satisfies conditions 52(13, 14), hence $(A + B) = (I + BQ)A$, since $QA = I$. The alternativity (13) implies the Moufang identities in the algebra R :

$$(M1) \quad (XYX)Z = X(Y(XZ)),$$

$$(M2) \quad Z(XYX) = ((ZX)Y)X,$$

$$(M3) \quad (XY)(ZX) = X(YZ)X \text{ for each } X, Y, Z \in R.$$

From Lemma 53 and Formulas (M1, M2) it follows that $C(A + B) = (Q(I - BQ + (BQ)^2 - \dots))((I + BQ)A) = (Q - Q(BQ) + Q(BQ)^2 - \dots)(A + B) = I$.

55. Corollary. *The set U_l of all left invertible elements $A \in R$ is an open subset in R .*

56. Notation. Denote by R' the algebra over \mathbf{R} of all periodic functions $x : [0, 2\pi] \rightarrow R_{\mathbf{C}_i}$ of the form

$$(1) \quad x(t) = \sum_{n=-\infty}^{\infty} a_n e^{nti}$$

with coefficients $a_n \in R$ such that

$$(2) \quad \sum_n \|a_n\| < \infty$$

with point-wise addition and multiplication of functions

$$(3) \quad (x + y)(t) = x(t) + y(t), \quad (xy)(t) = x(t)y(t) \text{ for each } t \in [0, 2\pi].$$

57. Lemma. *Suppose that $x \in R'$ and $x(0)$ has a left inverse in R , then there exists an element*

$$(1) \quad y(t) = \sum_{n=-\infty}^{\infty} c_n e^{nti} \in R'$$

such that

$$(2) \quad c_0 \text{ has a left inverse } q_0 \text{ in } R \text{ and}$$

$$(3) \quad \|q_0\| \sum_{n=1}^{\infty} \|c_n + c_{-n}\| < 1 \text{ and}$$

$$(4) \quad \text{there exists } \epsilon > 0 \text{ so that } y(t) = x(t) \text{ for each } t \in (-\epsilon, \epsilon).$$

Proof. Consider the following function given piecewise $w_\epsilon(t) = 1$ for $|t| < \epsilon$, $w_\epsilon(t) = 2 - |t|/\epsilon$ for $\epsilon \leq |t| < 2\epsilon$, $w_\epsilon(t) = 0$ for $2\epsilon \leq |t|$, where $0 < \epsilon \leq \pi/2$. Then one defines the function

$$y_\epsilon(t) = w_\epsilon(t)x(t) + [1 - w_\epsilon(t)]x(0) = \sum_{n=-\infty}^{\infty} b_n(\epsilon)e^{nti}.$$

This function satisfies Condition (4). It has the Fourier series with coefficients $b_n = b_n(\epsilon)$:

$$b_n = \frac{3\epsilon}{2\pi}a_n + \sum_{k=1}^{\infty} \frac{a_{n-k} + a_{n+k}}{\pi k^2 \epsilon} (\cos(\epsilon k) - \cos(2\epsilon k)) - \sum_{k=-\infty}^{\infty} a_k \frac{\cos(\epsilon n) - \cos(2\epsilon n)}{\pi n^2 \epsilon}$$

for $n \neq 0$ and

$$b_0 = a_0 + \sum_{k=1}^{\infty} (a_{-k} + a_k) \left[1 + \frac{\cos(\epsilon k) - \cos(2\epsilon k)}{\pi k^2 \epsilon} - \frac{3\epsilon}{2\pi} \right].$$

Therefore,

$$\lim_{\epsilon \downarrow 0} \|b_0\| = \left\| \sum_{k=-\infty}^{\infty} a_k \right\| = \|y(0)\| > 0 \text{ and}$$

$$\sum_{k=1}^{\infty} [\|b_k\| + \|b_{-k}\|] \leq \sum_{k=-\infty}^{\infty} \|a_k\| A_k,$$

where a positive number $\delta > 0$ exists such that $0 \leq A_k < \epsilon^{1/2}[2|k|C + 9/\pi]$ for each $0 < \epsilon < \delta$ and every $k \in \mathbf{Z}$, where $C = \text{const} > 0$. Thus a positive number $\epsilon_0 > 0$ exists so that

$$\|b_0\| > \sum_{k=1}^{\infty} [\|b_k\| + \|b_{-k}\|]$$

for each $0 < \epsilon < \epsilon_0$ (see also [2, 33]). From Lemma 54 statements (2, 3) of this lemma follow.

58. Corollary. *If $y \in R'$ and $y(t)$ satisfies Properties (2, 3) of Lemma 57, then y has a left inverse z in R' .*

59. Theorem. *If R' is an algebra of periodic functions as in §56. Then $x(t)$ has a left inverse in R' if $x(t_0)$ has a left inverse in R for each t_0 .*

Proof. In view of Lemma 57 and Corollary 58 for each $\tau \in [0, 2\pi]$ a positive number $\epsilon > 0$ and an element $y_\tau \in R'$ exist such that $y_\tau(t)x(t) = I$ for each $t \in (\tau - \epsilon, \tau + \epsilon)$. The segment $[-\pi, \pi]$ is compact, that is, each its open covering has a finite subcovering, consequently, a finite number of functions y_τ induces a function $y \in R'$ so that $y(t)x(t) = I$ for each t .

60. Lemma. Suppose that $-\pi < \alpha < a < b < \beta < \pi$ and $x_1, x_2 \in R'$ and $x_2(t)$ has a left inverse for each $t \in (\alpha, \beta)$ and $x_1(t)$ vanishes for every $-\pi \leq t < a$ and $b < t \leq \pi$. Then an element $x_3 \in R'$ exists vanishing on $[-\pi, \alpha) \cup (\beta, \pi]$ such that

$$(1) \quad x_1(t) = x_3(t)x_2(t) \text{ for each } t \in [-\pi, \pi].$$

Proof. From Lemma 57 and Corollary 58 it follows, that to any $\tau \in [a, b]$ a positive number $\epsilon > 0$ and an element $y_\tau \in R'$ correspond such that $y_\tau(t)x_2(t) = I$ for each $t \in (\tau - \epsilon, \tau + \epsilon)$. As in §59 one gets that an element $z \in R'$ exists such that $z(t)x_2(t) = I$ for every $t \in [a, b]$. Put $x_3(t) = x_1(t)z(t)$, consequently, $x_3(t) = 0$ for each $t \in [-\pi, \alpha] \cup [\beta, \pi]$ and $x_3 \in R'$ and hence Assertion (1) is valid, since the algebra R satisfies Conditions 52(13, 14) and e^{nti} commutes with R and $(a_n e^{nti})(b_k e^{kti}) = (a_n b_k) e^{(n+k)ti}$ for each $a_n, b_k \in R$ and $n, k \in \mathbf{Z}$.

61. Lemma. Suppose that $x(t)$ is strongly integrable on $(-\pi, \pi)$ function with values in R and vanishes on $(-\pi, -\pi + \epsilon) \cup (\pi - \epsilon, \pi)$ with $0 < \epsilon < \pi/2$ and

$$(1) \quad f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\tau) e^{-\tau ti} d\tau,$$

$$(2) \quad a_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\tau) e^{-n\tau i} d\tau,$$

then

$$(3) \quad \int_{-\infty}^{\infty} \|f(t)\| dt < \infty$$

if and only if

$$(4) \quad \sum_{n=-\infty}^{\infty} \|a_n\| < \infty.$$

Proof. Consider a positive number $0 < \delta < \pi/2$ so that $x(t)$ vanishes on $[-\pi, -\pi + 2\delta) \cup (\pi - 2\delta, \pi)$. Put $\phi(t) = 1$ for $|t| < \pi - \delta$, $\phi(t) = \frac{\pi - |t|}{\delta}$ for $|\pi - \delta| \leq |t| < \pi$, $\phi(t) = 0$ for $\pi \leq |t|$. If

$$(5) \quad x(t) = \sum_{n=-\infty}^{\infty} a_n e^{nti},$$

then

$$(6) \quad x(t) = x(t)\phi(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{\tau ti} \left[\sum_{n=-\infty}^{\infty} a_n \frac{\cos(\tau + n)(\pi - \delta) - \cos(\tau + n)\pi}{(\tau + n)^2 \epsilon} \right] d\tau,$$

since this integral and this sum are absolutely convergent. Therefore, the function

$$(7) \quad f(t) := \sqrt{\frac{2}{\pi}} \left[\sum_{n=-\infty}^{\infty} a_n \frac{\cos(\tau + n)(\pi - \delta) - \cos(\tau + n)\pi}{(\tau + n)^2 \epsilon} \right]$$

satisfies Conditions (1, 3), if (4) is fulfilled.

Vise versa, Condition (3) implies that

$$(8) \quad \sum_{n=-\infty}^{\infty} \left\| \int_{n-1/2}^{n+1/2} f(t) dt \right\| < \infty,$$

consequently,

$$(9) \quad \int_{n-1/2}^{n+1/2} f(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x(t) \frac{2 \sin(t/2)}{t} e^{nti} dt.$$

Thus the Fourier series of the function $x(t) \frac{2 \sin(t/2)}{t}$ converges absolutely. Moreover, the Fourier series of the mapping $\frac{t}{2 \sin(t/2)} \phi(t)$ also is absolutely convergent. Thus the Fourier series of $x(t) = [x(t) \frac{2 \sin(t/2)}{t}] [\frac{t}{2 \sin(t/2)} \phi(t)]$ is absolutely convergent, since R' is an algebra over the real field \mathbf{R} and \mathbf{i} commutes with each $y \in R'$.

62. Corollary. *Let g and $f \in L(R)$, let also*

$$(1) \quad x_1(t) = \int_{-\infty}^{\infty} g(\tau) e^{-\tau ti} d\tau$$

vanish outside some interval $(a, b) \subset (-\pi, \pi)$. Suppose that

$$(2) \quad x_2(t) = \int_{-\infty}^{\infty} f(\tau) e^{-\tau ti} d\tau$$

is zero outside an interval $(\alpha, \beta) \subset (-\pi, \pi)$ with $\alpha < a$ and $b < \beta$ and $x_2(t)$ has a left inverse for each $\alpha < t < \beta$. Then an element $y \in L(R)$ exists so that

$$(3) \quad g(t) = \int_{-\infty}^{\infty} y(\tau) f(t - \tau) d\tau.$$

This follows immediately from Lemmas 60 and 61.

63. Lemma. *If $f \in L(R)$, then*

$$(1) \quad \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \|f(t + \epsilon) - f(t)\| dt = 0.$$

Proof. The Lebesgue measure is σ -finite and σ -additive, so the statement of this theorem for step functions is evident. In $L(R)$ the set of step functions

$$g(t) = \sum_{k=1}^n a_k \chi_{B_k}(t)$$

is dense with $a_k \in R$, $B_k \in \mathcal{B}(\mathbf{R})$, $n \in \mathbf{N}$, where $\mathcal{B}(\mathbf{R})$ denotes the σ -algebra of all Borel subsets of \mathbf{R} .

64. Lemma. Suppose that $f \in L(R)$ and $h \in L(R)$ so that $\text{supp}(h) \subset (-\epsilon, \epsilon)$ for some positive number $0 < \epsilon < \infty$. Then

$$\begin{aligned} (1) \quad & \int_{-\infty}^{\infty} \|f(t) \int_{-\infty}^{\infty} h(\tau) d\tau - \int_{-\infty}^{\infty} f(t+\tau) h(\tau) d\tau\| dt \\ & \leq \left[\int_{-\infty}^{\infty} \|h(\tau)\| d\tau \right] \sup_{|u| \leq \epsilon} \int_{-\infty}^{\infty} \|f(t+u) - f(t)\| dt. \end{aligned}$$

Proof. This follows from Fubini's theorem

$$\begin{aligned} & \int_{-\infty}^{\infty} \|f(t) \int_{-\infty}^{\infty} h(\tau) d\tau - \int_{-\infty}^{\infty} f(t+\tau) h(\tau) d\tau\| dt \\ & \leq \int_{-\infty}^{\infty} \|f(t) - f(t+\tau)\| \|h(\tau)\| d\tau dt \\ & \leq \int_{-\infty}^{\infty} \|h(\tau)\| d\tau \sup_{|u| \leq \epsilon} \int_{-\infty}^{\infty} \|f(t+u) - f(t)\| dt. \end{aligned}$$

65. Lemma. If $f \in L(R)$, then

$$(1) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \|f(t) - \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t+\tau) \frac{\sin^2(n\tau)}{\tau^2} d\tau\| dt = 0.$$

Proof. Put $h_0(t) = \frac{\sin^2(nt)}{t^2} = h_1(t) + h_2(t)$ with $h_1(t) = h_0(t)[1 - |t|\sqrt{n}]$ for $|t| \leq n^{-1/2}$, while $h_1(t) = 0$ for $|t| > n^{-1/2}$. An application of Lemmas 63 and 64 leads to

$$(2) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \|f(t) - \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t+\tau) h_1(\tau) d\tau\| dt = 0$$

and

$$\begin{aligned} (3) \quad & \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \frac{1}{\pi n} \|f(t+\tau) h_2(\tau) d\tau\| dt \\ & \leq \left[\int_{-\infty}^{\infty} \|f(t)\| dt \right] \lim_{n \rightarrow \infty} \frac{1}{\pi n} \int_{-\infty}^{\infty} h_2(\tau) d\tau. \end{aligned}$$

On the other hand,

$$(4) \quad \frac{1}{\pi n} \int_{-\infty}^{\infty} h_2(t) dt =$$

$$\begin{aligned} & \frac{1}{\pi n} \left[\int_{-\infty}^{\infty} \frac{\sin^2(nt)}{t^2} dt - \int_{-n^{-1/2}}^{n^{-1/2}} (1 - |t|\sqrt{n}) \frac{\sin^2(nt)}{t^2} dt \right] \\ &= \frac{2}{\pi} \int_{n^{1/2}}^{\infty} \frac{\sin^2(t)}{t^2} dt + \frac{2}{\pi\sqrt{n}} \int_0^{\sqrt{n}} \frac{\sin^2(t)}{t^2} dt = O(n^{-1/2} \ln n), \end{aligned}$$

consequently,

$$(5) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left\| \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t + \tau) h_2(\tau) d\tau \right\| dt = 0.$$

66. Theorem. Let $f \in L(R)$ and let the Fourier transform

$$(1) \quad x(\tau) = \int_{-\infty}^{\infty} f(t) e^{\tau ti} dt$$

have a left inverse in R for each $\tau \in [-\pi, \pi]$. Then the \mathbf{R} -linear combinations

$$(2) \quad \sum_n b_n f(t - \tau_n) \text{ with } b_n \in R$$

are dense in $L(R)$, where $\tau_n \in [-\pi, \pi]$.

Proof. Lemma 65 means that a function

$$(3) \quad f_\delta(t) = \frac{1}{\pi n} \int_{-\infty}^{\infty} f(t + \tau) \frac{\sin^2(n\tau)}{\tau^2} d\tau$$

exists so that

$$(4) \quad \int_{-\infty}^{\infty} \|f(t) - f_\delta(t)\| dt < \delta,$$

where $0 < \delta$. Consider its Fourier transform:

$$(5) \quad h_1(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{uti} \left[\frac{1}{\pi n} \int_{-\infty}^{\infty} f(t + \tau) \frac{\sin^2(n\tau)}{\tau^2} d\tau \right] dt$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{tui} \left[\frac{1}{\pi n} \int_{-\infty}^{\infty} \frac{\sin^2(n\tau)}{\tau^2} e^{-u\tau i} d\tau \right] dt, \text{ consequently,}$$

$$h_1(u) = \left(1 - \frac{|u|}{2n}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{uti} dt \text{ when } |u| < 2n$$

and $h_1(u) = 0$ for $|u| \geq 2n$. Analogously the Fourier transform

$$h_2(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{uti} \left[\frac{1}{2\pi n} \int_{-\infty}^{\infty} f(t + \tau) \frac{\sin^2(2n\tau)}{\tau^2} d\tau \right] dt$$

vanishes for each u so that $|u| > 4n$. The Fourier series of h_1 and h_2 over $(-8n, 8n)$ converge absolutely by Lemma 61. Then one can write $h_1(u) = h_2(u)h_3(u)$, where

$$h_3(u) = \int_{-\infty}^{\infty} \psi(t) e^{tui} dt$$

with $\psi \in L(R)$, since the algebra R is alternative and e^{tui} commutes with any $y \in R'$ for each real numbers t and u . Therefore, we deduce that

$$(6) \quad \int_{-\infty}^{\infty} f_{\delta}(t) e^{tui} dt = \int_{-\infty}^{\infty} \frac{e^{tui}}{2\pi n} \left\{ \left[\int_{-\infty}^{\infty} f(t+\tau) \frac{\sin^2(2n\tau)}{\tau^2} \right] \int_{-\infty}^{\infty} \psi(x) e^{uxi} dx \right\} d\tau dt, \text{ consequently,}$$

$$\int_{-\infty}^{\infty} e^{tui} \left[f_{\delta}(t) - \frac{1}{2\pi n} \int_{-\infty}^{\infty} f(t+x) \left\{ \int_{-\infty}^{\infty} \frac{\sin^2(2n\tau)}{\tau^2} \right\} \psi(\tau-x) d\tau \right] dx dt = 0$$

for each u , hence

$$(7) \quad f_{\delta}(t) = \frac{1}{2\pi n} \int_{-\infty}^{\infty} f(t+x) \Phi(x) dx, \text{ where}$$

$$\Phi(x) = \int_{-\infty}^{\infty} \frac{\sin^2(2n\tau)}{\tau^2} \psi(\tau-x) d\tau$$

is absolutely integrable. Lemmas 63 and 64 imply that

$$(8) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} f(t+x) \Phi(x) dx - \sum_{k=-n^2}^{n^2-1} f\left(t+\frac{k}{n}\right) \int_{k/n}^{(k+1)/n} \Phi(x) dx \right\| dt = 0.$$

From Formulas (5, 6) and Lemma 65 the assertion of this theorem follows.

67. Lemma. *Let R be an algebra with unit (see §52) and let \mathcal{I} be a maximal left ideal. Suppose that X is an additive group of all cosets R/\mathcal{I} and H is an algebra of homomorphisms of X onto itself produced by multiplying by elements of R from the left. Then X is irreducible relative to H .*

Proof. Consider the quotient mapping $\theta : R \rightarrow R/\mathcal{I}$ (see also §I.2.39 [28]). A space X is \mathbf{R} -linear, since R is an algebra over \mathbf{R} . Therefore, $\theta(R)a =: V_a$ is an \mathbf{R} -linear space for each nonzero element $a \in R \setminus \{0\}$. Put $S_a = \theta^{-1}(V_a)$. Evidently, S_a is a left ideal in R and $\mathcal{I} \subset S_a$, $S_a \neq \mathcal{I}$. The ideal \mathcal{I} is maximal, hence $S_a = R$, consequently, $V_a = X$ and $Ha = X$ for each nonzero element $a \in R \setminus \{0\}$.

68. Lemma. *Let R , \mathcal{I} , X and H be the same as in Lemma 67. Suppose that for a marked element $x \in R$ and each maximal left ideal \mathcal{I} the corresponding element $\theta(x)$ is left invertible in H . Then x is left invertible in R .*

Proof. An algebra R has the unit element $I \in R$. Each left ideal is contained in a maximal left ideal. Therefore, an element $x \in R$ is left invertible if and only if this element x is not contained in a maximal left

ideal. On the other hand, if $yx = I$, then $\theta(y)\theta(x) = \theta(I)$. Take cosets b_I and b_0 in R/\mathcal{I} so that $I \in b_I$ and $0 \in b_0$. Then $\theta(y)\theta(x)b_I = \theta(I)b_I$. But $\theta(I)b_I = b_I$, since $II = I$. If $x \in b_0 = I$, then $\theta(x)b_I = b_0$ and $\theta(y)\theta(x)b_I = b_0$, since $x \in \theta(x)b_I$.

69. Lemma. *A maximal left or right ideal \mathcal{I} in R is closed.*

Proof. If \mathcal{I} is not closed, then its closure $cl_R(\mathcal{I})$ in R contains an ideal \mathcal{I} . On the other hand, $cl_R(\mathcal{I})$ is a left or right ideal in R respectively, since R is a topological algebra. Therefore, $cl_R(\mathcal{I}) = \mathcal{I}$, since \mathcal{I} is maximal.

70. Lemma. *If R is a Banach algebra, and \mathcal{I} , X and H have the same meaning as in Lemma 67, then X is a Banach space, H is a normed algebra with norm $\|\theta(z)\|$ on H so that $\|\theta(x)\|_H \leq \|x\|_R$ for each $x \in R$. If moreover R is a Hilbert algebra over either the quaternion skew field or the octonion algebra \mathcal{A}_v with $2 \leq v \leq 3$, then X is a Hilbert space over \mathcal{A}_v .*

Proof. The quotient algebra R/\mathcal{I} is supplied with the quotient norm $\|\theta(x)\|_X = \inf_{z \in \theta(x)} \|z\|_R$ (see also §I.2.39 [28]). Therefore,

$$\|\theta(x)\|_H = \sup_{b \in X} \|\theta(x)b\|_X / \|b\|_X \leq \|x\|_R.$$

If R is a Hilbert algebra over \mathcal{A}_v , then $\|x\|_R = \sqrt{\langle x; x \rangle}$, where a scalar \mathcal{A}_v -valued product $\langle x; y \rangle$ on R satisfies conditions of §I.2.3 [28]. From the parallelogram identity and the polarization formula one gets that $\|\theta(x)\|_X$ induces an \mathcal{A}_v -valued scalar product on X (see Formulas I.2.3(1 – 3, SP) [28]).

71. Notation. Let R be a quasi-commutative C^* -algebra over either the quaternion skew field or the octonion algebra \mathcal{A}_r , $2 \leq r \leq 3$, satisfying conditions of §52. Let also F be a normed algebra of functions either $f, g : \Lambda \rightarrow \mathcal{A}_r$ or $(\mathcal{A}_r)_{\mathbf{C}_i}$ with point-wise multiplication $f(t)g(t)$ and addition $f(t) + g(t)$ of functions and $\phi : F \rightarrow \mathcal{A}_r$ or $(\mathcal{A}_r)_{\mathbf{C}_i}$ respectively be a continuous \mathbf{R} homogeneous additive multiplicative homomorphism, $\phi(fg) = \phi(f)\phi(g)$. Suppose that the unit function $h(t) = 1$ for each $t \in \Lambda$ belongs to F , also R' is a family of functions $x : \Lambda \rightarrow R$ or $x : \Lambda \rightarrow R_{\mathbf{C}_i}$ satisfying the following conditions:

(1) R' is an algebra over the real field \mathbf{R} under point-wise multiplication and addition;

- (2) if $x_1, \dots, x_n \in R$ and $f_1, \dots, f_n \in F$, then $(x_1 f_1 + \dots + x_n f_n) \in R'$;
- (3) R' is a normed algebra so that $\|x^f\| = \|x\| \|f\|$ in the case over \mathcal{A}_r or $\|x^f\| \leq \|x\| \|f\|$ over $(\mathcal{A}_r)_{\mathbf{C}_i}$ for each $x \in R$ and $f \in F$, where $x^f := x f$, $\|f\|$ denotes a norm of f in F ;

(4) the \mathbf{R} -linear combinations of Form (2) are dense in R' ;

(5) If $x = x_1^{f_1} + \dots + x_n^{f_n}$ and ϕ is a continuous homomorphism as above, then $\|x_1 \phi(f_1) + \dots + x_n \phi(f_n)\| \leq \|x\|$.

Each homomorphism ϕ of F induces $\hat{\phi}(x)$ with the property $\hat{\phi}(x^f) = x \phi(f)$ and $\hat{\phi}$ will be called a generated homomorphism.

72. Theorem. *Let suppositions of §71 be satisfied. Then an element $x \in R'$ has a left inverse if for each generated homomorphism $\hat{\phi}$ the corresponding element $\hat{\phi}(x)$ of R has a left inverse in R .*

Proof. Since a homomorphism ϕ is \mathbf{R} -homogeneous and additive, then it is \mathbf{R} -linear. Take an arbitrary maximal ideal \mathcal{I} in R' . It has the decomposition

$$(1) \mathcal{I} = \bigoplus_{j=0}^{2^r-1} \mathcal{I}_j i_j,$$

where \mathcal{I}_j is either a real or complex algebra isomorphic with \mathcal{I}_k for each $0 \leq j, k \leq 2^r - 1$. Each $x \in R'$ has the corresponding element $\theta(x)$ of H (applying Lemma 68 to R' here instead of R in §68).

The algebra R' has the decomposition $R' = R'_0 i_0 \oplus \dots \oplus R'_m i_m$ induced by that of R with pairwise isomorphic commutative algebras R'_j and R'_k either over \mathbf{R} or \mathbf{C}_i respectively for each k, j , $m = 2^r - 1$. Thus any two elements $a, b \in R'$ quasi-commute and $a = a_0 i_0 + \dots + a_m i_m$ and $b = b_0 i_0 + \dots + b_m i_m$ with $a_j, b_j \in R'_j$ for each j . Particularly, elements $I^f = I f$ of R' quasi-commute with each $x^g \in R'$ and hence with each $b \in R'$. In view of Theorem I.2.81 and Corollary I.2.84 [28] and Lemmas 67 and 70 above the mapping $\theta(I^f)$ is the continuous algebraic homomorphism from F into \mathcal{A}_r or $(\mathcal{A}_r)_{\mathbf{C}_i}$ correspondingly. There exists a homomorphism ϕ so that $\theta(I^f) = J \phi(f)$, where $J := \theta(I1)$ is a unit of H . Therefore, $\hat{\phi}(x_1^{f_1} + \dots + x_n^{f_n}) = \theta(x_1 1) \theta(I f_1) + \dots + \theta(x_n 1) \theta(I f_n) = \theta(x_1 1) \phi(f_1) + \dots + \theta(x_n 1) \phi(f_n) = \theta([x_1 \phi(f_1) + \dots + x_n \phi(f_n)]1)$, consequently, $\theta(x) = \theta(\hat{\phi}(x)1)$ for each $x = (x_1 f_1 + \dots + x_n f_n) \in R'$. From Condition 71(5) and Lemma 70 it follows that $\theta(x) = \theta(\hat{\phi}(x)1)$ for each $x \in R'$.

If $\hat{\phi}(x(t))$ has a left inverse y in R , then $\theta(y1)\theta(x) = \theta(yx)$, consequently, $\theta(yx) = \theta(\hat{\phi}(yx)1) = \theta(y\hat{\phi}(x)1) = \theta(I1) = J$. This means that $\theta(x)$ has a left inverse, hence by Lemma 68 x has a left inverse in R' .

73. Corollary. *If suppositions of §71 are fulfilled and for each $\phi(f)$ with $f \in F$ a point t_0 exists so that $\phi(f) = f(t_0)$, then Condition 71(5) can be replaced by $\|x(t_0)\| \leq \|x\|$ for each t_0 . Moreover, in the latter situation an element $x \in R'$ has a left inverse in R' , if $x(t)$ has a left inverse in either R or $R_{\mathbf{C}_i}$ correspondingly for each t .*

74. Remark. If an algebra F has not a unit, then one can formally adjunct a unit 1 and consider an algebra $\bar{F} := \{c1 + f : c \in Q, f \in F\}$, where either $Q = \mathcal{A}_r$ or $Q = (\mathcal{A}_r)_{\mathbf{C}_i}$ correspondingly, putting $|c1 + f|^2 = |c|^2 + |f|^2$ and $\bar{R}' := \{z = cI1 + x : x \in R', c \in Q\}$ with $\|z\|^2 = |c|^2 + \|x\|^2$. This standard construction induces an extended homomorphism either $\bar{\phi}(c1 + f) = c + \phi(f)$ or an exceptional homomorphism $\bar{\phi}(c1 + f) = c$. If F has not a unit, then statements above can be applied to \bar{F} and \bar{R}' so that an element $\bar{x} = cI1 + x$ with $c \neq 0$ may have a left inverse of the form $(bI1 + y)$.

75. Corollary. *Suppose that Γ is an additive discrete group so that $\Gamma = \Gamma_0 i_0 \oplus \dots \oplus \Gamma_m i_m$ with pairwise isomorphic commutative groups Γ_j and Γ_k for each $0 \leq j, k \leq m$ with $m = 2^r - 1$, $0 \leq r \leq 3$, while $G = G_0 i_0 \oplus \dots \oplus G_m i_m$ is an additive group so that G_j is dual to Γ_j with continuous characters $\chi(\beta, t) = \prod_{k=0}^m \chi_k(\beta_k, t_k) \in S^1$, where $S^1 := \{u \in \mathbf{C}_i : |u| = 1\}$, $\beta = \beta_0 i_0 + \dots + \beta_m i_m$ and $t = t_0 i_0 + \dots + t_m i_m$ with $\beta_k \in \Gamma_k$ and $t_k \in G_k$ for each k . Let*

$$(1) \quad x(t) = \sum_{\beta \in \Gamma} a_\beta \chi(\beta, t)$$

with $a_\beta \in R$ for each $\beta \in \Gamma$ and

$$(2) \quad \sum_{\beta \in \Gamma} \|a_\beta\| < \infty,$$

then $x \in R'$ has a left inverse in R' if $x(t)$ has a left inverse in $R_{\mathbf{C}_i}$ for each $t \in G$.

76. Corollary. *Let suppositions of Corollary 75 be satisfied, but with (2) replaced by*

$$(1) \quad \sum_{\beta \in \Gamma} e^{q(\beta)} \|a_\beta\| < \infty,$$

where $q(\beta) \in \mathbf{R}$ and

(2) $q(\alpha + \beta) \leq q(\alpha) + q(\beta)$ and $q(0) = 0$. Then $x \in R'$ is invertible, if $[\sum_{\beta \in \Gamma} a_\beta e^{p(\beta)} \chi(\beta, t)]$ has a left inverse in $R_{\mathbf{C}_i}$ for each $t \in G$ with a system of reals $p(\beta) \in \mathbf{R}$ so that

(3) $p(\alpha + \beta) = p(\alpha) + p(\beta)$ and $p(0) = 0$ and $p(\beta) \leq q(\beta)$ for each $\beta \in \Gamma$.

77. Corollary. Let suppositions of Corollary 76 be satisfied, but let Γ be a locally compact group with a nontrivial nonnegative Haar measure λ . If R' is formed by elements of the form:

$$(1) \quad x(t) = \int_{\Gamma} a_\beta \chi(\beta, t) \lambda(d\beta)$$

with $a_\beta \in R$ and

$$(2) \quad \int_{\Gamma} e^{q(\beta)} \|a_\beta\| \lambda(d\beta) < \infty.$$

If R' has the unit $1(t) = 1$ for each $t \in G$, then x is left invertible, if $x(t)$ has a left inverse in $R_{\mathbf{C}_i}$ for each $t \in \Gamma$. If R' has not a unit, but 1 is an adjoint unit as in §74, then an element $\bar{x} = c1 + x$ with $c \neq 0$ has a left inverse of the form $b1 + y$, if $[cI + \int_{\Gamma} a_\beta e^{p(\beta)} \chi(\beta, t) \lambda(d\beta)]$ has a left inverse for every $t \in G$ and each continuous system $p(\beta)$ satisfying Conditions 76(2, 3).

78. Remark. Duality theory for locally compact groups is contained in [31, 13]. Particularly, \mathcal{A}_r can be considered as the additive commutative group $(\mathcal{A}_r, +)$. As the additive group it is isomorphic with \mathbf{R}^{2^r} . The group of characters of \mathbf{R}^n is isomorphic with \mathbf{R}^n for any natural number n (see §23.27(f) in Chapter 6 of the book [13]). The Lebesgue measure on the real shadow \mathbf{R}^{2^r} induces the Lebesgue measure λ on \mathcal{A}_r , which is the Haar measure on $(\mathcal{A}_r, +)$ (see also §1).

It is possible to consider a dense subgroup K of the total compact dual group G , when Γ is discrete. It is sufficient an existence of a left inverse $y(t)$ of $x(t)$ for each $t \in K$ and that $\sup_{t \in K} \|y(t)\| < \infty$ due to the following lemma.

79. Lemma. Let x_n tend to x in R , when a natural number n tends to the infinity, let also y_n be a left inverse of x_n for each n and $\sup_n \|y_n\| < \infty$, then $x \in R$ possesses a left inverse.

Proof. From the equality $I - y_n x = I - y_n x_n + y_n x_n - y_n x$ it follows that $\|I - y_n x\| \leq \|y_n\| \|x_n - x\|$. Then Lemma 53 implies that a natural number k

exists so that $y_n x$ has a left inverse z_n for each $n \geq k$, consequently, $z_n y_n$ is a left inverse of x due to the alternativity of the algebra R or using Moufang's identities.

80. Corollaries. *Suppose that an algebra R is over the Cayley-Dickson algebra \mathcal{A}_v (see §52) and $0 \leq r \leq v$ and $2 \leq v \leq 3$, $\Gamma = (\mathcal{A}_r, +)$ (see §§75-78).*

(1). *If*

$$x(t) = \sum_n a_n e^{(\beta(n), t) \mathbf{i}} \in R'$$

with $a_n \in R$ and $\sum_n \|a_n\| < \infty$, $(\beta, t) = Re(\beta t^) = \beta_0 t_0 + \dots + \beta_m t_m$, where $\beta = \beta_0 i_0 + \dots + \beta_m i_m \in (\mathcal{A}_r, +)$, $t = t_0 i_0 + \dots + t_m i_m \in G = (\mathcal{A}_r, +)$, $m = 2^r - 1$, moreover, a left inverse $z(t)$ exists for each t and $\sup_t \|z(t)\| = C < \infty$, then a function*

$$y(t) = \sum_n b_n e^{(\tau(n), t) \mathbf{i}} \in R'$$

exists with $\sum_n \|b_n\| < \infty$ such that $y(t)x(t) = I$ for each t .

(2). *If $a(\beta) \in R$ is a strongly integrable function with*

$$\int_{\mathcal{A}_r} \|a(\beta)\| \lambda(d\beta) < \infty$$

and if for a nonzero complex number $c \in \mathbf{C}_i \setminus \{0\}$ a function

$$[cI + \int_{\mathcal{A}_r} a(\beta) e^{(\beta, t) \mathbf{i}} \lambda(d\beta)]$$

has a left inverse for all t (see λ in §78), then a left inverse of the form

$$[qI + \int_{\mathcal{A}_r} b(\beta) e^{(\beta, t) \mathbf{i}} \lambda(d\beta)]$$

exists with

$$\int_{\mathcal{A}_r} \|b(\beta)\| \lambda(d\beta) < \infty.$$

81. Corollary. *The algebra $L_q^{n, per}(l_\infty(\mathbf{Z}, Y))$ is the saturated subalgebra in the algebra $L_q^c(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over either the quaternion skew field or the octonion algebra \mathcal{A}_v , $2 \leq v \leq 3$.*

82. Theorem. *Suppose that $B \in L_q^{n, per}(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over either the quaternion skew field or the octonion algebra \mathcal{A}_v , $2 \leq v \leq 3$. Then the following conditions are equivalent:*

- (1) an operator B is invertible;
- (2) a Fourier transform operator $\hat{B}(M)$ is invertible for each $M \in \mathbf{S}^1$ and $x \in Y$.

Proof. A real linear Banach subspace X_k is considered, which is linearly isometrically isomorphic with $l_\infty(\mathbf{Z}, Y_k)$ for each $k \geq 0$. On the other hand, the real span $\text{span}_{\mathbf{R}}\{x \in X_k i_k : k \geq 0\}$ is dense in X . In view of Theorem 51 an operator B is uniformly c -continuous, $B \in L_q^{uc}(l_\infty(\mathbf{Z}, Y))$. Therefore, its invertibility on $\text{span}_{\mathbf{R}}\{x \in X_k i_k : k \geq 0\}$ is equivalent to that of on X .

Let $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$ be an invertible operator with $D = B^{-1}$. In view of Corollary 81 the inverse operator D is n -periodic as well and has an absolutely converging Fourier series. From Theorem 51 it follows that D is uniformly c -continuous. Applying Corollary 47 and Proposition 45 we deduce that the Fourier transform operator \hat{B} is invertible for each $M \in \mathbf{S}^1$ and $x \in Y$, that is $\hat{B}(M)\hat{D}(M)x = \widehat{BD}(M)x = I_Y x = x$. Thus (1) \Rightarrow (2).

Vise versa suppose that Condition (2) is fulfilled. Then by Corollary 81 the mapping $\psi : M \mapsto (\hat{B}(M))^{-1}$ has an absolutely converging Fourier series:

$$(1) \quad \psi(M) = \sum_{k=-\infty}^{\infty} M^k {}_k D, \text{ where}$$

$$(2) \quad {}_k D := \frac{1}{2\pi} \int_0^{2\pi} e^{-itk} \psi(e^{it}) dt \in L_q(Y \oplus Y\mathbf{i}).$$

Put $D = \sum_{k=-\infty}^{\infty} {}_k \bar{D} S(k)$, where as usually $S(k)$ denotes the coordinatewise shift operator on k , $S(k)x(l) = x(l+k)$, while ${}_k \bar{D}$ denotes an 1-ribbon operator, matrix elements of which on the k -th diagonal are equal to ${}_k D$. Therefore, the operator D is bounded with $\|D\| \leq \sum_{k=-\infty}^{\infty} \|{}_k D\| = c < \infty$. In accordance with Corollary 47 the operator D is inverse of B , i.e. $D = B^{-1}$.

83. Corollary. Let $B \in L_q^{n,per}(l_\infty(\mathbf{Z}, Y))$, where Y is a Banach space over either the quaternion skew field or the octonion algebra \mathcal{A}_v , $2 \leq v \leq 3$. Then spectral sets of \mathbf{B} and $\mathbf{B}(D(\mathcal{M}))$ are related by the formula:

$\sigma(\mathbf{B}) = \bigcup_{M \in \mathbf{S}^1} \sigma(\mathbf{B}(D(\mathcal{M})))$, where \mathbf{B} denotes the natural extension of B from $l_\infty(\mathbf{Z}, Y)$ onto $l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i})$.

Proof. The spectral set $\sigma(\mathbf{B})$ is the complement of the resolvent set (see Definition I.2.6 [28]), where $Y \oplus Y\mathbf{i}$ and $l_\infty(\mathbf{Z}, Y \oplus Y\mathbf{i})$ have structures of \mathcal{A}_v

Banach spaces as well. In view of Proposition 45 and Theorem 82 one gets this corollary.

84. Theorem. *Let a kernel K of a periodic operator B from §1 satisfy the condition:*

$$(1) \quad \sup_{t,s} \|K(t,s)\| = c_1 < \infty,$$

where $2 \leq v \leq 3$. Then an operator $A = I - B$ is invertible if and only if a Fourier transform operator $\hat{A}(M)$ is invertible on $Y \oplus Y\mathbf{i}$ for each $M \in \mathbf{S}^1$.

Proof. Condition (1) implies that an operator B is bounded and the integral of §1 exists due to the theorem in §1 (see also Chapter 10 in [7]). Take an operator $A = I - B \in L_q(L^p(\mathcal{A}_w, Y))$, where $p \in [1, \infty]$. Choose a domain V in the Cayley-Dickson algebra \mathcal{A}_w so that $V = \{z : z \in \mathcal{A}_w; \forall j \ z_j \in [0, \omega_j]; z = \sum_{j=0}^{2^w-1} z_j i_j\}$. Then we define an operator $U : L^p(\mathcal{A}_w, Y) \rightarrow l_p(\mathbf{Z}^{2^w}, L^p(V, Y))$ by the formula: $((Ux)(t))(\bar{m}) := y_{\bar{m}}(\tau)$, where $x \in L^p(\mathcal{A}_w, Y)$ and $y \in l_p(\mathbf{Z}^{2^w}, L^p(V, Y))$ are related by the equation: $y_{\bar{m}}(\tau) = x(t - \sum_j m_j \omega_j i_j)$ with $\tau_j = t_j - m_j \omega_j \in [0, \omega_j]$ for every $0 \leq j \leq 2^w - 1$, where $\bar{m} = (m_0, \dots, m^{2^w-1}) \in \mathbf{Z}^{2^w}$. This definition implies that such operator U is an invertible isometry.

There exists an operator $Q = UAU^{-1}$, hence $Q \in L_q(l_p(\mathbf{Z}^{2^w}, L^p(V, Y)))$. Evidently Q is invertible if and only if A is such. If $S(\omega)x(t) = x(t + \omega)$ and $\hat{S}(\bar{m})y_{\bar{n}} = y_{\bar{m}+\bar{n}}$ are shift operators, they satisfy the equation $\hat{S}(\bar{m}) = US(\sum_j m_j \omega_j i_j)U^{-1}$. Therefore, the operator Q commutes with each shift operator $\hat{S}(\bar{m})$, where $\bar{m} \in \mathbf{Z}^{2^w}$. Thus, this operator Q is 1-periodic by each $m_j \in \mathbf{Z}$ (see Definition 33).

For a function $x \in L^p(V, Y)$ put $\bar{x}_{\bar{m}}(\bar{k}) := [\prod_{j=0}^{2^w-1} \delta_{m_j, k_j}]x$, hence $\bar{x}_{\bar{m}} \in l_p(\mathbf{Z}^{2^w}, L^p(V, Y))$. By each variable m_j a matrix of the operator Q takes the form: $Q_{k_j, s_j}x = (Q\bar{x}_{\bar{s}})(\bar{k}) = (UAU^{-1}\bar{x}_{\bar{s}})(\bar{k}) = (UAy_{\bar{s}})(\bar{k})$, when $k_l = s_l$ for each $l \neq j$, where $y_{\bar{s}} \in l_p(\mathbf{Z}^{2^w}, L^p(V, Y))$ is given by the formula: $y_{\bar{s}}(\tau) = 0$ if there exists j so that $\tau_j \notin [s_j \omega_j, (s_j + 1)\omega_j)$, while $y_{\bar{s}}(\tau) = x(t - \sum_j s_j \omega_j i_j)$ when $\tau_j \in [s_j \omega_j, (s_j + 1)\omega_j)$ for each j , where $\tau = t - \sum_j s_j \omega_j i_j$. This means that a tensor operator $Q_{\bar{k}}^{\bar{s}}$ is defined: $Q_{\bar{k}}^{\bar{s}} = (Q\bar{x}_{\bar{s}})(\bar{k})$. Then the function $(UAy_{\bar{s}})(\bar{k}) \in L^p(V, Y)$ takes the form:

$$(2) \quad (UAy_{\bar{s}})(\bar{k})(\tau) = x(\tau)$$

$$\begin{aligned}
& -\sigma \int_{\gamma^\alpha(b)|_{b_0 \in [s_0 \omega_0, (s_0+1)\omega_0]}} \cdots \sigma \int_{\gamma^\alpha(b)|_{b_u \in [s_u \omega_u, (s_u+1)\omega_u]}} K((\tau + \sum_j k_j \omega_j i_j), b) \\
& \quad x(b - \sum_j s_j \omega_j i_j) db_0 \dots db_u \\
& = x(\tau) - \sigma \int_{\gamma^\alpha(b)|_{b_0 \in [s_0 \omega_0, (s_0+1)\omega_0]}} \cdots \sigma \int_{\gamma^\alpha(b)|_{b_u \in [s_u \omega_u, (s_u+1)\omega_u]}} K((\tau + \sum_j k_j \omega_j i_j), \\
& \quad (b + \sum_j s_j \omega_j i_j)) x(b) db_0 \dots db_u \\
& = x(\tau) - \sigma \int_{\gamma^\alpha(b)|_{b_0 \in [s_0 \omega_0, (s_0+1)\omega_0]}} \cdots \sigma \int_{\gamma^\alpha(b)|_{b_u \in [s_u \omega_u, (s_u+1)\omega_u]}} K(\tau, \\
& \quad b + \sum_j (s_j - k_j) \omega_j i_j) x(b) db_0 \dots db_u
\end{aligned}$$

in accordance with Conditions 1(5, 6), where $u := 2^w - 1$. This implies that a tensor operator takes the form:

$$\begin{aligned}
(3) \quad Q_{\bar{s}}^{\bar{k}} & = x(\tau) - \sigma \int_{\gamma^\alpha(b)|_{b_0 \in [s_0 \omega_0, (s_0+1)\omega_0]}} \cdots \sigma \int_{\gamma^\alpha(b)|_{b_u \in [s_u \omega_u, (s_u+1)\omega_u]}} K(\tau, \\
& \quad b + \sum_j (s_j - k_j) \omega_j i_j) x(b) db_0 \dots db_u
\end{aligned}$$

for each $\bar{s}, \bar{k} \in \mathbf{Z}^{2^w}$. Elements of this tensor depend only on the difference $\bar{s} - \bar{k}$, so it is possible to put $Q_{\bar{s}}^{\bar{k}} = Q_{\bar{s}-\bar{k}}$.

The operators U , A and U^{-1} are c -continuous by each s_j , consequently, the operator $Q_{\bar{s}}$ is also c -continuous by each s_j , where $j = 0, 1, \dots, 2^w - 1$. Applying Theorem 82 we obtain the statement of this theorem.

85. Corollary. *Let an operator B satisfy conditions of Theorem 84, then a spectral set is $\sigma(\mathbf{Q}) = \bigcup_{M \in \mathbf{S}^1} \sigma(\mathbf{Q}(D(\mathcal{M})))$, where \mathbf{Q} denotes the natural extension of Q from $l_\infty(\mathbf{Z}^{2^w}, L^\infty(V, Y))$ onto $l_\infty(\mathbf{Z}^{2^w}, L^\infty(V, Y \oplus Y\mathbf{i}))$.*

Proof. This follows from Theorem 84 and Corollary 83 applying the Fourier transform by each variable, since $\mathbf{Q} \in L_q^{1,per}(l_\infty(\mathbf{Z}^{2^w}, L^\infty(V, Y) \oplus L^\infty(V, Y)\mathbf{i}))$ due to Condition 84(1) and the latter Banach space over the Cayley-Dickson algebra \mathcal{A}_v is isomorphic with $L_q^{1,per}(l_\infty(\mathbf{Z}^{2^w}, L^\infty(V, Y \oplus Y\mathbf{i})))$.

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